Trajectory-Based Quantum Chaos*

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Abstract

Chaotic behavior of a nonlinear system is characterized by the high sensitivity of its phase-space trajectories to initial conditions. Because the trajectory of a particle is not a well-defined quantity in quantum mechanics, the manifestations of chaotic motion in quantum mechanics remain a controversial issue. This difficulty now can be resolved by employing the nonlinear dynamics derived from complex mechanics.

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Chaotic behavior of a nonlinear system is characterized by the high sensitivity of its phase-space trajectories to initial conditions. Because the trajectory of a particle is not a well-defined quantity in quantum mechanics, the manifestations of chaotic motion in quantum mechanics remain a controversial issue. To retain the meaning of trajectory sensitivity in quantum chaos, the de Broglie-Bohm's causal interpretation of quantum mechanics [1] is a promising approach. However, the de Broglie-Bohm's trajectories are defined in a real space, but according to the El Naschie's $E^{(\infty)}$ theory [2,3], the topological structure of the quantum world is indeed characterized by a 4-dimensional complex spacetime. Hence, a full description of quantum trajectory becomes possible only within a complex space. In complex mechanics [4,5], quantum chaos is analyzed with trajectory defined in the complex spacetime. By this complex extension, a broader condition from which quantum chaos emerges can be gained, and chaos in many quantum systems can be identified, which otherwise was considered impossible using de Broglie-Bohm approach.

In complex mechanics, a particle moving on a complex plane is described by

$$x = x_R + i x_I \in \mathbb{C}, \qquad (1)$$

with $x_R \in \mathbb{R}$ and $x_I \in \mathbb{R}$ being the real and imaginary part of x, respectively. Analogously, its momentum is described by $p = p_R + i p_I \in \mathbb{C}$. The complex equation of motion for a particle with mass m reads [4]

$$m\frac{dx}{dt} = \frac{\hbar}{i}\frac{\partial}{\partial x}\ln\Psi(t,x) = \frac{\hbar}{i\Psi(t,x)}\frac{\partial\Psi(t,x)}{\partial x}, \quad x \in \mathbb{C}. \qquad (2)$$

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The total energy of the system is given by

$$H(t,x,p) = \frac{1}{2m}p^2 + V(t,x) + Q(\Psi(t,x)), \tag{3}$$

where the quantum potential

$$Q(\Psi(t,x)) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\ln\Psi(t,x). \tag{4}$$

is the origin of quantum chaos. Equation (2) is a non-autonomous complex differential equation and is equivalent to two non-autonomous real differential equations by the decomposition (1):

$$\frac{dx_R}{dt} = \mathrm{Re}\left(\frac{\hbar}{im\Psi}\frac{\partial\Psi(t,x)}{\partial x}\right) = f_R(x_R,x_I,t), \quad \frac{dx_I}{dt} = \mathrm{Im}\left(\frac{\hbar}{im\Psi}\frac{\partial\Psi(t,x)}{\partial x}\right) = f_I(x_R,x_I,t). \tag{5}$$

In reality, only the real part x_R can be measured, but the interaction between x_R and x_I via the coupling equations in Eq. (5) produces the chaos behavior observed from $x_R(t)$. Under the framework of complex mechanics, we thus need three autonomous equations to describe a 1D quantum system by treating time as an additional state variable in Eq. (5). According to the Poincaré-Bendixson theorem, now it is possible to find chaotic behavior from Eq. (5). By comparison, only two autonomous equations are required to describe a 1D quantum system by using de Broglie-Bohm formulation, for which chaotic behavior is impossible.

For a given initial position, $x(0) = x_R(0) + ix_I(0)$, a unique trajectory can be found from Eq. (5) on the complex plane. However, the uniqueness of trajectory on the complex plane does not imply the uniqueness of trajectory on the real axis, since different points on the complex plane may be projected into the same point on the real axis. By fixing the real-part initial position $x_R(0) = x_R^0$ and letting $x_I(0)$ vary in $\mathbb{R}$, a set of complex trajectories can be determined from Eq. (5):

$$\Omega = \left\{(x_R(t), x_I(t)) \,\middle|\, \dot{x}_R = f_R(x_R,x_I,t),\ \dot{x}_I = f_I(x_R,x_I,t),\ x_R(0) = x_R^0,\ x_I(0) \in \mathbb{R}\right\}. \tag{6}$$

The projection of Ω into the real space gives rise to a set of real trajectories,

$$\Omega_R = \left\{x_R(t) \,\middle|\, \dot{x}_R = f_R(x_R,x_I,t),\ \dot{x}_I = f_I(x_R,x_I,t),\ x_R(0) = x_R^0,\ x_I(0) \in \mathbb{R}\right\}, \tag{7}$$

as if they all originate from the same initial position $x_R(0) = x_R^0$. Since only the real part $x_R(t)$ can be measured, what we observe are the trajectories in Ω_R, which comprises infinite number of real trajectories all emerging from the same initial position $x_R(0) = x_R^0$. This is just the multi-path phenomenon considered in Feynman's fractal space-time approach to quantum mechanics [6].

Classical chaos analysis is to consider the sensitivity of $x_R(t)$ to the perturbation of the initial position $x_R^0 \to x_R^0 + dx_R^0$. One of the distinct features of quantum chaos is that even if the initial perturbation dx_R^0 is zero, the trajectory $x_R(t)$ still perturbs and diverges. We call this chaos phenomenon strong chaos [7], because the chaos behavior is so severe that the trajectory diverges spontaneously without any perturbation in the initial position. All of such perturbed trajectories with $dx_R^0 = 0$ are caused by the unobservable $x_I(0)$ as shown in Eq. (6), and are included in the set Ω_R. The analysis of strong chaos then amounts to considering the trajectory divergence in Ω_R due to the variation in the value of $x_I(0)$.

One-dimensional harmonic oscillator was found to have no chaos behavior by both the classical and quantum-mechanical treatments. However, if we take into account its complex motions, we will see that chaos does occur in 1D harmonic oscillator. The eigenfunction for the Schrödinger equation with potential $V(x) = kx^2/2$ is given by

$$\Psi_n(\bar{x},\bar{t}) = C_n H_n(\bar{x})\exp(-\bar{x}^2)\exp(-i(n+1/2)\bar{t}),\ n = 0, 1, 2, \ldots , \tag{8}$$

where $\bar{x} = (\sqrt{mk}/\hbar)^{1/2}x$, $\bar{t} = t\sqrt{k/m}$ and $H_n(\bar{x})$ is the nth-order Hermite polynomial. Quantum trajectories solved from Eq. (5) with wavefunction Ψ given by Eq. (8) show that quantum motions in pure eigenstates are periodic with quantized periods of oscillation. The superposition of at least two eigenstates is a necessary requirement to exhibit chaos in a 1D quantum system. As an example, we consider the superposition of four eigenstates:

$$\begin{aligned}\Psi(\bar{x},\bar{t}) = \exp(-\bar{x}^2/2)\big[&a_0\exp(-i\bar{t}/2) + a_1\cdot 2\bar{x}\exp(-3i\bar{t}/2)\\ &+ a_2(4\bar{x}^2-2)\exp(-5i\bar{t}/2) + a_5(32\bar{x}^5 - 160\bar{x}^3 + 120\bar{x})\exp(-11i\bar{t}/2)\big]\end{aligned} \tag{9}$$

where the coefficients $a_0, a_1, a_2, a_5 \in \mathbb{C}$ represent the relative magnitudes of the four eigenstates corresponding to $n = 0$, 1, 2, and 5. Chaos in other types of superposition can be analyzed in the same way. For the case of $a_0 = 1$, $a_1 = a_2 = 0$, and $a_5 = 0.1$, the corresponding motion is described by the following equation:

$$\dot{\bar{x}} = f(\bar{x}) = \mathrm{i}\bar{x} - \mathrm{i}\frac{80\bar{x}^4 - 240\bar{x}^2 + 60}{16\bar{x}^5 - 80\bar{x}^3 + 60\bar{x} + 5e^{5i\bar{t}}}\ . \tag{10}$$

Setting initial position to $\bar{x}_0 = 0.01 + 0i$, we obtain a wandering trajectory around certain regions on the complex plane as shown in Fig. 1a. The broad band of the frequency spectrum of $\bar{x}_R(\bar{t})$ depicted in Fig. 1b is a typical signature of chaos. The Poincaré section of system (10) is constructed by recording the data point $(\dot{\bar{x}}_R, \bar{x}_R)$ when the complex trajectory crosses $\bar{x}_I = 0$. The distribution of $(\dot{\bar{x}}_R, \bar{x}_R)$ shown in Fig. 1c clearly signifies the chaotic behavior appearing in the 1D quantum system described by Eq. (10).

It is noteworthy that a 1D classical harmonic oscillator subjected to the potential $V(x) = kx^2/2$ alone does not exhibit chaos. The driving force to invoke chaos in a 1D quantum harmonic oscillator is the extra quantum potential defined in Eq. (4). Substituting the wavefunction (9) with $a_0 = 1$, $a_1 = a_2 = 0$, and $a_5 = 0.1$ into Eq. (4), we obtain the quantum potential Q in this entangled state as

$$Q(\bar{x},\bar{t}) = \frac{25e^{10i\bar{t}} + 40\bar{x}(75 - 60\bar{x}^2 + 4\bar{x}^4)e^{5i\bar{t}} + 400(9 + 9\bar{x}^2 + 8\bar{x}^6) - 256\bar{x}^8(5 - \bar{x}^2)}{2(16\bar{x}^5 - 80\bar{x}^3 + 60\bar{x} + 5e^{5i\bar{t}})^2}\ . \tag{11}$$

Fig. 1d plots the real part of the total potential combining the classical potential $V(\bar{x})$ and the quantum potential $Q(\bar{x},\bar{t})$. Due to the time-varying nature of $Q(\bar{x},\bar{t})$, only the pattern of the total potential at the instant $\bar{t} = 0$ is shown in Fig. 1d, where we can see that a series of sharp rises in the quantum potential occurs along the x_I axis. If a particle moves into these regions full of cliffy potential, it may be driven to an entirely different direction due to a very slight perturbation in its position, which then gives rise to the observed chaotic behavior as shown in Fig. 1a. On the other hand, a particle moving far away from the x_I axis will experience no quantum potential and will follow a regular trajectory. Consequently, the reason why initial positions within some regions will produce chaotic motion, while those in other regions will produce regular motion can be explained by the action and the distribution of the quantum potential Q.

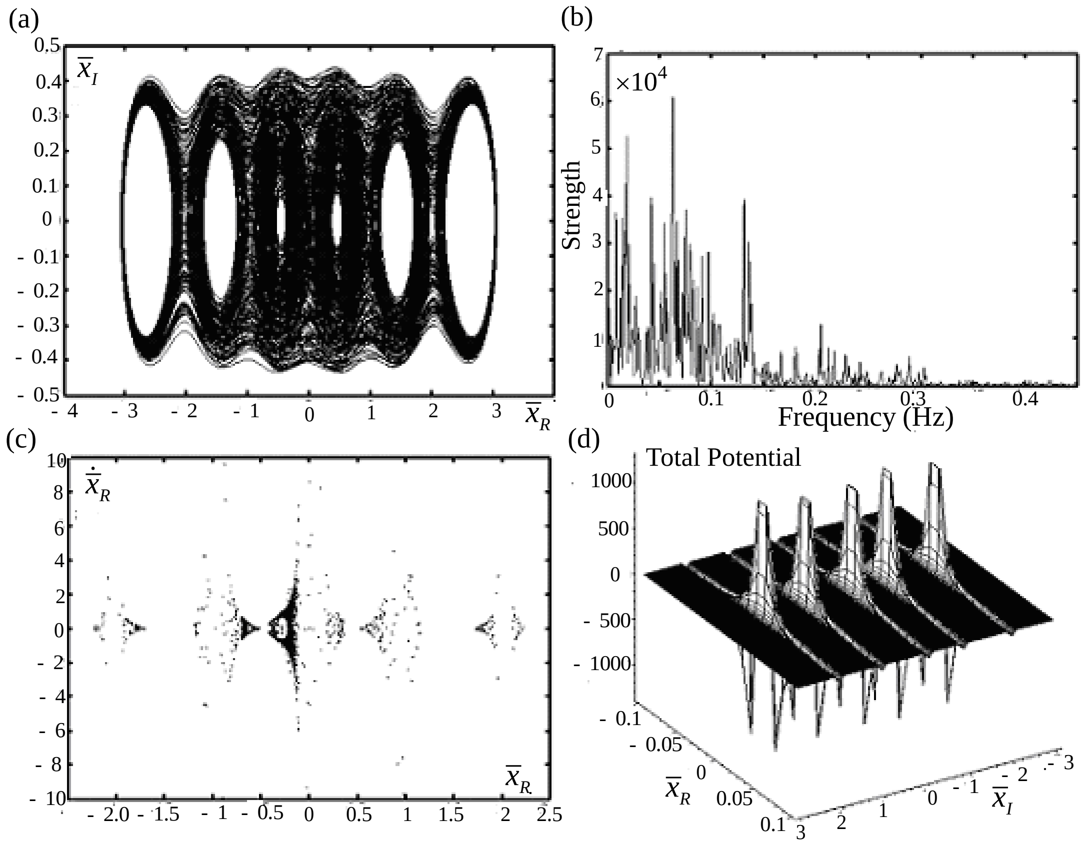

Fig. 1 Four signatures of chaos for the chaotic system described by Eq. (9) with $a_0 = 1, a_5 = 0.1$, and $a_1 = a_2 = 0$.

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