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The theory of Cantorian spacetime and high energy particle physics (an informal review)

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A R T I C L E I N F O

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A B S T R A C T

The paper gives a rather detailed introduction and up to date review of the main concepts and ideas upon which the theory of fractal-Cantorian spacetime is based.

1. Introduction

Einstein's Relativity Theory was a revolutionary departure from our habitual classical picture of a mechanical processes taking place on a passive spacetime stage uninfluenced by it and vice versa [1]. It may be likened to the difference between the classical theatre of Racine and a modern play by Luigi Pirandello.

While Einstein's relativity-theory has shown us that, in the large, spacetime geometry is curved and quite different from our passive, flat and smooth classical space and time, modern results in high energy physics are forcing us to reconsider the properties not only of flatness but also of the smoothness of the geometry of spacetime at the quantum scale [2,3]. This is exactly what we are proposing in this paper. We will present a substantially novel approach to quantum gravity and particle physics based on the idea that spacetime is basically a large infinite-dimensional but hierarchical, disconnected and thus non-differentiable Cantor Set [4,5].

1.1. *Clash of symmetries*

To meet the special requirements of particle physics as well as gravity many excellent unification spacetime theories have been proposed with varying degrees of success [6,7]. A crucial stumbling block of a consistent revision of Euclidean spacetime topology and geometry is the potential and frequently detrimental clash between classical spacetime symmetry in which our experiments are inevitably conducted and the internal Gauge symmetry required by particle physics. Such a clash of symmetries leads to what is known in quantum field theory as anomalies [8,9]. It turns out that there are two strategies to completely eliminate these anomalies. The first possibility is to formulate our geometry and topology without any direct reference to the concept of points. A well-known example for this type of theory is Connes' non-commutative geometry which is a natural extension of von Neumann's continuous geometry [10,11]. In fact, von Neumann used to joke about his own proposal by calling it point-less geometry.

The second possibility is to avoid having any points at all in the theory. This possibility was made use of in String theory and one of the main reasons for the phenomenal success and popularity of String-Theory lies in this fact [3]. Without excluding points the anomaly cancellation procedure of Green and Schwarz leading to a ten dimensional super symmetric spacetime theory, replacing the old 26-dimensional Bosonic String theory would not have been possible [3].

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1.2. Fractal spacetime and Cantor sets

In our work which began about two decades ago, we started exploring the possibility of a geometry which in a sense reconciles the irreconcilable namely having points which are no points in the ordinary sense [4,12]. In other words we could have our cake and eat it by using geometry with points which upon close examination reveal themselves not as a point but as a cluster of points. Every point in this cluster, when re-examined, reveals itself again as another cluster of points and so on ad infinitum. Not surprisingly this type of geometry is well-known since a long time to mathematicians and seems to have been discovered first by the German mathematician Georg Cantor, the inventor of Set-Theory [13,14]. The famous Triadic Cantor Set is probably the simplest and definitely the best known example of such geometry (see Fig. 1). Cantor sets are at the heart of modern mathematics and a particular form of non-metric spaces and geometry, known in modern parlance as fractals. However, Cantor Sets have never been used explicitly to model spacetime in physics. Material scientists, mechanical engineers, chemists, and biologists, all apart of the hard-core non-linear dynamicists apply Cantor Sets across the fields [15]. By contrast it seems indeed that our group is the first to take the idea of a Cantorian fractal spacetime seriously and give it a viable mathematical formulation with the help of which specific and precise computations can be made [16–57]. It is this approach that we wish to introduce in this paper.

1.3. Basic assumptions

Starting from the basic assumption that spacetime is essentially a very large Cantor Set we are going to extract from this simple theory a great deal of information about particle physics, gravity and the combination of the two, namely quantum gravity [21].

We recall that gravity is a property of spacetime in the large. By contrast, particle physics is a property of spacetime at the very small, or more accurately on the quantum level of observation. This disparity in size which corresponds to disparity in the energy scale is one, if not the main reasons behind the unyielding resistance against all attempts to reconcile the two fundamental theories. However, in Cantorian fractal spacetime where unlike the smooth classical case there is no a-priori given natural scale it is very easy to make the very large meet the very small exactly as in P-Adic number theory [22]. In this sense we can speak of a coincidentia oppositorium, to use a Hegelian terminology. In theoretical physics this is what is called by Edward Witten T-duality in connection with M-theory [23]. In other words, a Cantorian fractal spacetime has an inbuilt T-duality or a P-Adic property as well as being free of anomalies and scale invariant all for the simple reason of not having any ordinary points in it [24].

We will start in the next section by discussing in more detail how to construct the said Cantorian spacetime manifold from scratch as well as deriving its dimensionality and finally looks at the dynamics induced by it.

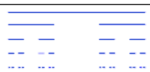
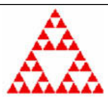
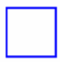
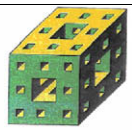
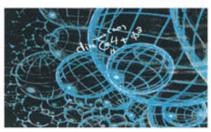
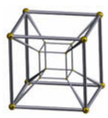
Type of fractal	Geometrical shape	Menger–Urysohn dimension	Hausdorff dimension	Corresponding random Hausdorff dimension	Embedding dimension	Corresponding Euclidean shape	Remark
Cantor Set		0	*ln 2/ln3 = 0.630929753*	$\phi = 0.61803398$	1	*Line*	The middle third of the line is removed and the iteration is repeated to obtain Cantor set. The final total length is zero.
Sierpinski gasket		2	*ln 3/ln2 = 1.584962501*	$\frac{1}{\phi} = 1.618033989$	2	 *Square*	Hausdorff dimension of this fractal is the inverse of the Hausdorff dimension of the classical cantor set.
Menger sponge		3	D_{MS} = *ln20/ln3 = 2.7268*	$2 + \phi = 2.61803398$	3	 *Cube*	The COBE temperature of microwave background radiation is found to be T_c(COBE) = D_{MS}K = 2.726 K.
The 4 dimension random cantor set analogue of Menger sponge	 *An artist impression of* $\varepsilon^{(\ )}$ *space-time*	4	$d_c^{(4)} =$ 4.236068	$4 + \phi^3 =$ 4.23606797	5	 *Hyper cube*	Note that E-infinity was not postulated but rather motivated by physical considerations. It was derived mathematically from first principles using set theory.

Fig. 1. Fractal–Hausdorff dimensions from E-infinity point of view. Notice that our classification and comparison with orderly classical Cantor set, Sierpinski gasket, Menger sponge and the four dimensional analogue of the Menger sponge and hypercube is almost complete. We just need the exact chaotic fractal shape of the fractal hypercube.

2. Constructing a Cantorian spacetime and deriving its dimensionality from basic first principles

The manifold corresponding to the Cantorian spacetime that we are proposing is strictly speaking non-differentiable and thus not a manifold. By definition a manifold is differentiable, however a Cantor Set is not continuous and therefore, strictly speaking, not a manifold. Nevertheless there are many important non-differentiable structures nowadays which we still regard as quasi-manifold and for the sake of simplicity we are dropping the word quasi. In fact, this property of not being continuous, is more than welcome in particle physics which obeys the Planck quantum. In such a case we do not have the schism between a discrete matter and energy field and a continuous spacetime which it inhabits. On the other hand general relativity requires a smooth and differentiable spacetime manifold linking gravity with the geometry of spacetime. In fact this is again the wonderful thing about our manifold. Looking at this manifold from afar, that is to say reducing the accuracy of observation which in turn is equivalent to reducing the energy scale, our manifold appears smooth and differentiable. In other words we have an observation dependent topology [25]. By putting parts of this smooth manifold under a magnifying glass, its real non-differentiable nature becomes apparent again. Not only that, but even the dimensionality of our manifold is also observation dependent, that is to say energy scale dependent. Looking at our manifold from very far, it appears four dimensional; but taking it under a magnifying glass the dimensionality may increase to 10, 26, 137, or even infinitely many dimensions if our magnifying glass is infinitely strong. In fact at such a point the T-duality becomes operative again and consequently the largest meet the smallest once more. This is neatly and accurately given by the remarkable P-Adic norm $\|\bar{\alpha}_o = 137\|_2 \equiv \bar{\alpha}_{QG} = 1$ where $\bar{\alpha}_o$ is the inverse electromagnetic fine structure constant and $\bar{\alpha}_{QG} = 1$ is the Planck coupling constant [23,24]. The same scale dependency also exists for the curvature of spacetime manifold and is therefore extended to the corresponding dynamics [see Table 1]. Let us see how we can formulate this scale dependency mathematically.

2.1. Mathematical formulation – the Hausdorff dimension of a Cantor set

As we said earlier on a Cantor Set is the simplest transfinite set that there is. Let us construct a Triadic Cantor Set (See Fig. 1). Consider the unit interval. Remove the middle third of this interval, except for the end points. We are now left with two intervals of the length one third. Remove the middle one third of each of the two left intervals except again for the end points. Repeating this process infinitely many times we are left with a remarkable completely disjointed points set. This point set has no length, because we have removed the entire interval. Mathematicians refer to that as a point set of measure zero. In some sense there is nothing left anymore. Miraculously, however, this alleged nothingness has a respectable and relatively sizable dimension. Such point set dimension is called a Hausdorff dimension, which in this case is equal to the natural logarithm of the number of the parts left in each iteration divided by the number of divisions in each iteration. In our case this is $\ell\mathbf{n}\ 2$ divided by $\ell\mathbf{n}\ 3$.

This is not the only wonder of this set. An additional mathematically stringent fact which sounds unbelievable, but is true, is that the number of points in our Cantor set is not one point less than the number of points in the original continuous line-interval. In both cases we have not only infinitely many points, but actually uncountable infinitely many points. The original expression for that was coined by Cantor who stated this fact simply as the equality of "Mächtigkeit" of the Cantor set and the continuum, whether one or multi dimensional. The common word used nowadays in English and German is cardinality [25]. It turns out that the Cantor set is a perfect compromise between the discrete and the continuum. Here we have a discrete structure, yet it has the same cardinality as the continuum, and here lies the secret of the success of our Cantorian spacetime model which unites the ununitable as we said earlier on [42–57].

2.2. Deriving the Hausdorff dimension of Cantorian spacetime

Now we take a Cantor set which is not necessarily a triadic one but is embedded in the space of a topological dimension one. The Hausdorff dimension will be denoted by $d^{(0)}$. The exact numerical value of this Hausdorff dimension which must be less than unity is left open and will be fixed as a result of our construction and not put in by hand. Let us add infinitely many such Cantor sets together. The first set Hausdorff dimension is $d^{(0)}$, the second will be $(d^{(0)})^2$, the third will be $(d^{(0)})^3$, and so on. For the power infinity the corresponding Hausdorff dimension will be zero summing $(d_c^{(0)})^n$ from zero to infinity gives us $1/(1 - d^{(0)})$. To be able to say how many dimensions there are in terms of the dimension of the original set we must divide the sum by $d^{(0)}$. Thus the final expression can be shown to be effectively a weighted Hausdorff dimension of the entire new

Table 1
The super string hierarchy obtained by scaling $(\bar{\alpha}_0/2)$ using ϕ^n for $n = 1$ to $n = 6$.

n	Dimension or inverse coupling	Interpretation
1	$42 + 2k$	$= (1/\alpha_g)$ Grand unification inverse coupling $= \bar{\alpha}_g$
2	$26 + k$	$= (1/\alpha_{gs})$ Super symmetric grand unification $= \bar{\alpha}_{gs}$ or the fractal Riemann-like curvature
3	$16 + k$	Extra bosonic dimension of Heterotic superstrings or the fractal Ricci-like curvature
4	10	the 10-dimensions of superstrings
5	$6 + k$	The compactified six dimensions of the Calibi-Yau manifold of superstrings
6	$4 - k$	spacetime dimension

set which is formally made of infinitely many dimensions, but each dimension has a different weight. Consequently although we have infinitely many dimensions we can still sum them all and the sum will converge to a finite value given by $1/[d^{(0)}(1-d^{(0)})]$ [25].

We could interpret what we have just done in a different way. We have in essence added infinitely many topological dimensions n from zero to infinity, but each dimension has a weight equal to $(d^{(0)})^{n}$. Consequently the sum of all dimensions is equal to the sum from 0 to infinity of $(n)\,(d^{(0)})^{n}$. This series is easily summed up exactly and is equal $(1+d^{(0)})/(1-d^{(0)})$. This expression may be regarded as the average topological dimensions where our infinite-dimensional but hierarchal Cantor set lives. By contrast the first sum was the average Hausdorff dimension. Since our Cantorian manifold is supposed to model spacetime it should not have gaps or overlapping points. We can achieve that by equating both expressions. By doing that we find an equation to determine $d^{(0)}$ from the condition that the dimensionality of the embedding space and the dimensionality of the manifold to be embedded in it should be equal. The mathematical terminology for that is "space filling" condition [25–27]. Proceeding in this way the resulting equation $(d^{(0)})^2+d^{(0)}-1=0$ is a simple quadratic equation with the only positive root $d_c^{(0)}=(\sqrt{5}-1)/2=0.618033989$, namely the Golden Mean [28].

Substituting this value back into our average dimensions we find a remarkable number for the average Hausdorff and embedding dimension, namely $4+\phi^3=4.236\,0679$. This is the Hausdorff dimension of the core of our Cantorian spacetime manifold as well as being the average topological dimension of the embedding manifold. In a moment we will demonstrate that the corresponding topological dimensionality which is now called Menger–Uhrysohn dimension is exactly 4 (see Fig. 1). This would be the first time ever that a mathematical derivation has been given for the dimensionality of spacetime from primitive mathematical and topological assumptions [4,5,25]. The reader should give attention to how the Menger–Uhryson and the Hausdorff dimension are linked in a single equation $d^{(n)}=(1/d_c^{(0)})^{n-1}$. For $n=0$ we find $d_c^{(0)}=\phi$ as should be. For $n=1$ we find our so called normality condition $d_c^{(1)}=1$. However only for $n=D=4$ we find our Hausdorff dimension $d(4)=(1/\phi)^{4-1}=(4/\phi)^3=4+\phi^3$. That is why we say the Menger–Uhrysohn topological dimension is exactly 4.

2.3. The deep meaning of the dimensionality of spacetime

Before talking about details, let us recapitulate what we have said and done so far. We have established a non-differentiable and non-metric topological space described not by one dimension but by three. Formally our spacetime manifold is infinite-dimensional, however it has two other finite dimensions, namely an average Hausdorff dimension of $4+(\phi)^3$ and in addition a topological Menger–Uhrysohn dimension which is exactly 4. That is why this manifold can mimic many other dimensionalities depending on the resolution of the observation. Said differently: that is why in this manifold there is no clash between internal dimensions and classical dimensions. Looked at it from very far it looks like a four dimensional manifold exactly as the spacetime of relativity. On closer examination, however, it is a four dimensional manifold surrounding another four dimensional manifold which is again enclosing a further four dimensional manifold and so on ad infinitum. This Russian doll picture is substantiated mathematically by the continuous fraction expansion of the Hausdorff dimension which is equal to $4+(\bar{4})$ with 1/4 repeated ad infinitum in the form of a continued fraction as shown in Fig. 2. In this figure we draw analogy between smooth spaces that is to say line, square, cube and higher-dimensional cube and their Cantorian counterpart which is a Cantor set, Sierpinski gasket, a Menger sponge and spacetime which is difficult to draw but for which we have just calculated the Hausdorff dimension and found it to be $4+\phi^3$. Please note that in Fig. 2 we have Cantor Sets, Sierpinski gasket and Menger sponge with the classical Hausdorff dimensions [29]. These are the three, two and one dimensional versions of our $4+\phi^3$, $(2+\phi)$, $(1+\phi)$ and ϕ. That means, what was in the classical case $\ell n2/\ell n3$ is now simply ϕ. The two numbers are close but not identical. Similarly $\ell n3/\ell n2$ of the Sierpinski is now $2+\phi=2.618033898$.

Since the Sierpinski gasket lives in two dimensions one is naturally inclined to suspect that the Sierpinski is the two dimensional generalization of the one dimensional Cantor set which is correct. However the Sierpinski is not an area. It is a two dimensional curve. The next natural question is to ask how to generalize this result to three and higher dimensions. Let us leave this question for the moment unanswered and go back to our Golden Mean Hausdorff dimension which we have just determined. What is the two dimensional version of this set? To answer this question in an elementary manner we proceed in the following way.

It is a well-known fact that the Golden Mean is the limit of the most important mathematical gross law which nature employs in countless processes, namely the Fibonacci gross law. In the Fibonacci series every term is the sum of the previous two terms. If we assume that our basic Cantor set which we just have seen has a Golden Mean Hausdorff dimension will grow like a drop of ink spreading in a fluid but in this case obeying the Fibonacci law, then we can formulate the following law for our Cantor set. The First term in our series will be the dimension we started with which is the golden mean. The second term will be set equal one. Now following Fibonacci, the third term must be the sum of the two previous terms which is $\phi+1$. This would be the Hausdorff dimension of the two dimensional version of our original Golden Mean Cantor set. Two nice surprises which are not surprises pop up. First: $1+\phi$ is roughly 1.618. This is quite close to the dimension of the Sierpinski gasket which is $\ell n3/\ell n2=1.585$. The second is that the reciprocal value of ϕ, namely $1/\phi$ is exactly equal to $1+\phi$. Thus our generalization to two dimensions is formally the same generalization from classical Cantor set to classical Sierpinski gasket [25].

To generalize to three dimensions we proceed in the same manner and add the previous two dimensions, but we have to order our series in the correct Fibonacci manner, namely, first ϕ which we give the well known deductive

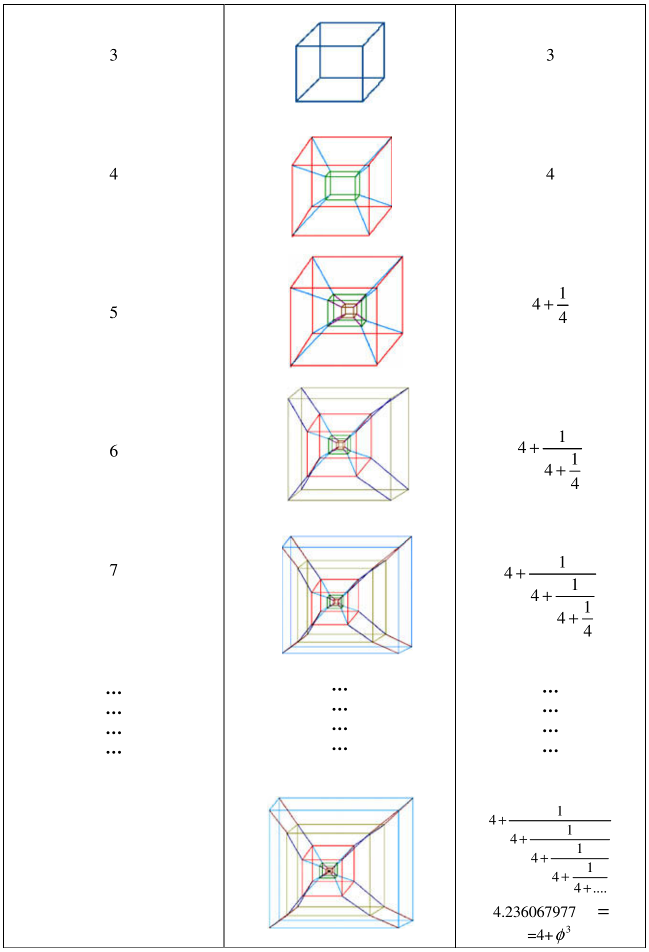


Fig. 2. Fractal dimensions of hypercube starting from a three dimensional cube. A process of cube nesting leads at infinity to the average of expectation value of the Hausdorff dimension of fractal-Cantorian spacetime $4 + \phi^3$ when $\phi = \frac{\sqrt{5}-1}{2}$. Thus although we have infinitely many dimensions we have a finite Hausdorff expectation dimension for the nested structure.

Menger–Uhrysohn dimension zero. Secondly the one which we give the Menger–Uhrysohn dimension 1. Third, $1 + \phi$ which we give the Menger–Uhrysohn dimension 2. Now the Menger–Uhrysohn dimension three gives simply $1 + (1 + \phi)$ as a Hausdorff dimension which is roughly 2.618. It is very nice to note that this three dimensional generalization of our set has a Hausdorff dimension very close indeed to the dimension of the Menger sponge which looks in essence like the three dimensional extension of the classical Sierpinski gasket. The Hausdorff dimension of the Menger sponge is $\ell n\, 20/\ell n 3 = 2.7268$. It is important to note however that the Menger sponge is not a 3D volume but rather a 3D curve with zero volume.

We come now to the most crucial point of this section and our theory: What will the four dimensional generalization of our Cantor set be? To answer the question we follow the Fibonacci gross law and add the previous two terms, which means $2 + \phi + 1 + \phi$. This miraculously adds up to exactly $4 + \phi^3$ in full agreement with our earlier deduction in Section 2.2 and demonstrates that the topological dimension corresponding to $4 + \phi^3$ is exactly 4. Strictly speaking we should not say topological dimensions but we should say Menger–Uhryson dimension instead (see Fig. 1). It is important to note that all the preceding results could have been obtained directly from the formula $d^{(n)} = (1/\phi)^{n-1}$ which we call the bijection formula and which we introduced in paragraph 2.2.

2.4. The three different dimensions fixing our Cantorian spacetime

The preceding derivation gave us a marvellous result which we interpret topologically as follows. We recall that we started with an infinite number of Cantor sets of various dimensions. When mixed and added together we obtained a total average dimension for the so formed manifold equal $4 + \phi^3$. This result was obtained by requiring that the average Hausdorff dimension and the average topological Menger–Uhryson dimension are the same, which turned out to be $4 + \phi^3$. In addition to this result, we just found that the Menger–Uhrysohn dimension, corresponding to $4 + \phi^3$ is exactly 4. This is the result we promised the reader to prove earlier on. Indeed, we have shown that we have a remarkable, non-differentiable manifold which we could call now Cantorian spacetime manifold which is fixed by three dimensions as we said earlier. It is formally infinite-dimensional, but seen from very far, it mimics our smooth four dimensional spacetime manifold, in addition to having a Hausdorff dimension equal $4 + \phi^3$. This we interpreted earlier on as a self-similar universe with four dimensions inside four dimensions, inside four dimensions and so on (see Fig. 2) like a Russian doll [25].

2.5. *The random nature of Cantorian spacetime*

Before closing this section we have to clarify a single point and make an important statement: What is the difference, if any, between the classical triadic Cantor Set and our Golden Mean Hausdorff dimension Cantor Set? It turns out that this is a crucial difference in principle. We recall, we did not specify at all how we actually constructed our Golden Mean Cantor Set. For our derivation this was not a problem, but for understanding the meaning this is extremely important. It was two American mathematicians, Mauldin and Williams, who actually constructed a one dimensional Golden Mean Cantor Set. They replaced the orderly triadic construction by a random construction. In their original paper they said they used a uniform probabilistic distribution which is the simplest assumption which one can make to introduce randomness to anything. Ergo by requiring space-filling we have de facto introduced randomness to our model. Space filling means: no gaps and no overlapping. This is an optimal situation reminiscent of the classical problem of densest sphere-packing. When you pack spheres of the same size, then we will clearly have gaps between the different spheres. To have these gaps filled, we have to fill them with densely packed spheres of smaller size and so on ad infinitum. The result is a kind of fractal sphere packing. From everyday experience we know that when we fill, say, a jar with different size marbles, then we have to shake this jar to get the densest packing. What we have really done is introduce randomness to get optimal density. This is the crucial difference between normal fractal structure and the fractal structure of our Cantorian spacetime manifold, which is inherently and implicitly random, as the analogy with Mauldin–Williams theorem shows a posteriori [25].

We claim that the preceding analysis constitutes the first mathematical derivation of the exact dimensions of spacetime starting from basic principles. In this derivation our Golden Mean Hausdorff dimension Cantor Set played the role of the elementary particle of spacetime, that is to say the building blocks of the medium which we frequently refer to wrongly as empty nothingness or esoteric, although we should know better at least since the advent of Einstein relativity which produces force and thus energy and matter from the curvature of spacetime, i.e. nothingness [52].

3. Average fractal symmetry and its dimensions

As we will see throughout this work one of the most important new ingredients which we introduced and which helped us to solve many problems is the irrationality inherent in the dimensions of our fundamental building blocks. Specifically this is the Golden Mean and its derivatives. As is well-known, the Golden Mean is the number which can be least approximated efficiently by a rational number. Therefore it is the most irrational number that can exist. This property of irrationality makes it play a fundamental role in KAM-theorem of Hamiltonian non-linear dynamics. An orbit with the Golden Mean as the winding number is dynamically the most stable one and in phase space representation is called Cantori. A Cantori is the last orbit to be destroyed by perturbation and is used therefore as a criterion for the onset of global chaos. This property will have consequences in our theory. For instance, in Euclidian quantum theory a great problem which arises is the inherent instability of the structure which this theory predicts. By contrast, the Golden Mean insures that the structure predicted by our theory is highly stable.

Not only that, but when we regard the Golden Mean 0.618033989 to be $0.5 + k_2$ where $k_2 = 0.118033989 = (1 + k)/10$, then the irrational k_2 plays a double role. It introduces fuzziness to our system and this fuzziness ensures a quasi-Hamiltonian behaviour in the phase space. In topology, quasi-Hamiltonian is called simplectic dynamic. This means preservation of the phase-space area which is indicative of a conservative system with no energy gain or losses. So we can see now the contradictory role of the Golden Mean: on the one side it preserves simplectic and Hamiltonian behaviour and on the other side it introduces a mathematical substitute for friction to ensure resilience and stability of the system. We have all these advantages at our disposal because of the marvellous mathematical device called the Hausdorff dimension. By extending the concept of topological dimension which is always an integer value to a Hausdorff or fractal dimension which is generically non-integer, that means it could be a rational or irrational number, we are able to describe things which we could not describe as easily before. For instance having the Golden Mean as the Hausdorff dimension we see that the Cantor Set that it described is more than a point but less than a line. It is something in between. In addition, the Hausdorff dimension could be used to describe the complexity of a geometrical object. Therefore, in essence, it is similar to entropy apart of having the meaning of a pseudo volume. There is no doubt that a Cantor Set is in general something more intricate at least on the surface of it than a straight smooth line. The topological dimension fails to capture the essence of a fractal [15,57]. By contrast the intrinsic Hausdorff dimension can do what the topological dimension cannot. The Menger–Uhrysohn dimension is of course a generalization of the topological dimension setting a Cantor set as a zero dimensional while -1 is the dimension of the empty set. Our theory generalizes the Menger–Uhrysohn system where the empty set is $-\infty$, rather than -1.

3.1. The dimension of average symmetry and the Gamma function

The Hausdorff dimension is not the first extension of a concept originally conceived to be an integer which has been generalized to non-integer. A famous example which happened to be very important is the extension of the factorial function $n!$ which is not defined for non-integer, to the Gamma function $\Gamma(n)$ which handles rational and irrational numbers as well as it handles integers. In all high energy particle physics theories one is forced invariably to consider symmetry groups. Most of the symmetry groups are Lie groups and theses groups have of course an integer dimension. However our non-differential Cantorian spacetime could not make efficient use of such groups. To be able to do that we should invent some Lie groups with non-integer dimensions which would fit better in our basic assumptions which we made ab initio about our manifold. Initially this proposition seems like a non-starter. However, on reflection it is easily reasoned that this is possible. A Lie group corresponds always to a Lie manifold. This manifold must have a dimension. Since we have expanded the concept of dimension from an integer topological one to a non-integer Hausdorff dimension, then we can do the same for a Lie manifold. It follows then logically that the corresponding Lie symmetry group could have a non-integer dimension. In fact, irrational small corrections to originally crisp Lie symmetry group dimensions would be a blessing for the overall model. We have mentioned earlier on in the Introduction that Gauge anomaly may be interpreted as the clash between different symmetries. With transfinite irrational corrections or fuzzy tail added or subtracted from the original crisp dimension, we obtain in essence fuzzy symmetry group dimensions which fit together harmoniously and eliminate gauge anomalies. In this connection we note on passing that a symmetry group has a volume which could be interpreted physically. For instance Vol (SO(3))=$8\pi^2$ = 78.95683521 is the 'tHooft action for an instanton [30]. We note on passing that the transfinitely exact value is equal Dim E6 + 3k = 78.54101966 where E6 is the exceptional Lie symmetry group and k = $\phi^3(1-\phi^3)$; $\phi = (\sqrt{5}-1)/2$.

3.2. The Golden Mean platonic solids

A simple example which illustrates our point regarding the vital role of the Golden Mean is the Penrose tiling of the plane. Without the Golden Mean proportionality of the different two parts used for Penrose tiling, nothing would fit at the end and we will have gaps or overlapping. Paradoxically, when the different lengths of the tiling parts are not fuzzy irrational numbers, nothing fits seamlessly. This brings us to the Platonic bodies. In three dimensions there are only five of them. The most important two are Golden Mean proportioned. Coxeter extended this to four dimensions and found his famous polytopes. Again these polytopes could not be constructed, not even in principle, without the proportionality of the Golden Mean. Making a large leap now, we can mention that the skeleton upon which the most important symmetry group used in superstring theory, namely the E8 exceptional Lie Group could be constructed from two 600 cells Coxeter polytopes by sliding a smaller one inside a larger one. The result is what we call E8 Gosset. In other words, the E8 exceptional Lie symmetry group, similar to all the five members of the exceptional family are based on Golden Mean proportioned Gossets [31]. Our theory is thus deeply related to the work of Luminet on well proportioned spacetime manifolds and his Poincaré Dodecahedron [58,59].

3.3. Transfinite corrections

To explain our procedure for producing the transfinitely corrected dimensions of the corresponding Lie groups the simplest way is to consider an example. The dimension SO(n) is given, as is well known, by Dim (SO(n) = (n)($n-1$)/2. For $n = 4$ the dimension is 6. Let us take n to be the Hausdorff dimension of our Cantorian spacetime manifold $n = 4 + \phi^3$. The dimension in this case is 6.854102. This is 0.854102 larger than the previous one. This transfinite tail is vital for certain calculations. For instance if we take twenty copies of the value of the Lie group dimension for $n = 4 + \phi^3$ then the total dimension will be 137.082 039. It can easily be shown that this value is the exact transfinite one for the inverse electromagnetic fine-structure constant. This relations would not be easily visible at all from setting $n = 4$.

Without going into the details of this theory which is beyond the scope of the present summary, we state that the exact transfinitely corrected value of the dimension of the E8 Lie exceptional symmetry group can be obtained from the preceding inverse electromagnetic fine-structure constant by a Golden Mean scale. For the crisp case the dimension of E8 is as well-known, 248. In the fuzzy transfinitely corrected case one finds

$$\text{Dim E8} = \frac{1}{2}(3+\phi)(137.082039325) = 247.983739 = 248 - k^2/2$$

where $k = \phi^3(1-\phi^3)$.

From this and similar reasoning the dimensions of the various involved symmetry groups may be interpreted not only as volumes, but also as energy gauged in appropriate units as is the case for the π Meson, the K Meson and η Meson [25].

One is now rightfully entitled to ask if we can explain the meaning of fuzzy dimensions of a Lie group in an intuitive way. Indeed we can: Fuzzy symmetry dimensions, that is to say, non-integer dimensions refer to average symmetries. This kind of average symmetry is a concept well known in the non-linear dynamics theory of strange attractors. As chaotic and complex as these strange attractors may be, they still posses a curious form of symmetry. In a sense this chaotic average symmetry is a higher form of symmetry than ordinary orderly symmetry. Maybe the three-point chaos game could illustrate the point. In this game which is nicely explained in [32] the essential point and the final conclusion is that when you proceed orderly you get no pattern and no symmetry. By contrast when you proceed randomly you get the Sierpinsky gasket in its full glory.

Similarly in Statistical Mechanics, when you make one picture at a time of a complex diffusion or convection experiment you see chaotic pattern, but when you superimpose hundreds of such pictures then an average symmetry and pattern begin to unfold. This is what is commonly labelled by the slogan symmetry of chaos [32]. In short the fuzzy dimension of the Lie symmetry group takes into account the implicit randomness of our Cantorian spacetime manifold and consequently it is best suited to describe it.

3.4. Feynman's path integral

In the next section we will try to explain how our procedure is similar to Feynman's path integral [33]. In this integral formulation of quantum mechanics which has become an indispensable tool in modern theoretical physics and quantum field theory, we sum over all paths. Every path has a weight and the more we deviate from the straight line connecting two points as a path for a classical particle the more we deviate from the classical description and come nearer to the behaviour of quantum-particles. The final result is given by a weighted average. We are not doing something very different in our fractal-Cantorian spacetime model from that because we are summing over dimensions instead of paths and giving every dimension a weight, all infinitely many of them as will be explained in the next section.

4. Summing over Lie groups

Feynman attempts to return at least half-heartedly to the notion of a path in quantum mechanics and to spacetime physics was to sum over all conceivable paths which a quantum particle can travel simultaneously. We constructed our Cantorian spacetime manifold by summing over infinitely many weighted dimensions which is quite similar. In the last few years the Author made a very important discovery which at the first moment may not seem as that much connected to our subject, but it is. This is explained in what follows.

4.1. Summing over Lie symmetry group dimensions

To explain the point we have to recall a few important facts: it is well-known that we have five exceptional Lie groups. These are E8, E7, E6, F4, and G2. The dimensions corresponding to these groups are 248, 133, 78, 52, and 14. Later on it was found that the unification group SU(5) with 24 dimensions and SO(10) with 45 dimensions may be regarded based on their Coxeter–Deykin diagram as E4 and E5, respectively [31]. In addition we were able to divide the well-known Lie group of the Standard Model in such a way as to represent lower-dimensional exceptional groups. Using various classification methods the Author was able to show that the total sum of dimension of the so obtained eight exceptional Lie groups is exactly 548. When taking transfinite corrections into account the total sum proved to be an invariant dimension exactly equal to 4 multiplied with the exact theoretical value of the inverse electromagnetic fine-structure constant, i.e. $4\bar{\alpha}_o = 548$. Finally this result could be interpreted as an energy content and a corresponding action [35]. This was the first discovery.

The second discovery was even more unexpected and is the following. There is surprisingly a finite number of two and three Stein-spaces, Einstein-space being the only one stein space that is not counted. Interestingly the number is exactly 17. Not one more and not one less. That should ring a bell, because in two dimensions there are only 17 distinct tiling patterns. These patterns are sometimes called Islamic or Arabic tiling and correspond exactly to seventeen symmetry groups which exhaust all possibilities [9,12,36]. These groups of symmetries are frequently called wall-paper groups or the two dimensional crystallographic groups. The surprising discovery which we made is that the total sum of dimensions of these groups is exactly 686. On close examination and including again transfinite corrections, the total sum was found to be exactly 5 multiplied with the inverse of the electromagnetic fine structure constant. The exact number is 685.410167 and corresponds again to energy and action principle [36]. In fact we were able to show that this result is equal to the square value of the fractal zero dimensional point curvature of our fractal spacetime manifold.

At this point we note that the multiply connected version of our three dimensional Euclidean space may be tiled by polyhedron. In this case we have 17 distinct multiply connected Euclidean spaces [58]. This 17 is reminiscent of the number of two dimensional crystalographic groups which we generalized to 219 three dimensional groups and then related them to Heterotic string theory to find the 8872 first level massless particle-like states.

4.2. Energy from dimensions and coupling constants

What is so useful about having conserved numbers such as 548 and 685 is that in essence we are holding infinity in the palm of our hand. Although finite the sum includes almost everything. In fact, we can show that it is the value of a corresponding action that is the integration of a Lagrangian. Before wondering about what this means, we fully understand the astonishment of the reader. This Lagrangian is indeed a scalar but we can obtain our pseudo-equation of dynamics by scaling this scalar-action using the main scaling exponent of our Cantorian manifold which is the Golden Mean. We will first have to show that this is indeed the internal energy of our manifold. This is for sure possible and turned out to be the square of the curvature of our manifold. Again, this curvature is of course a scalar similar to the Ricci scalar of general relativity and since our manifold unifies all fundamental forces, it is maybe not so surprising to know that the scalar-curvature is equal to the inverse super symmetric unification coupling constant. This was determined some time ago using the Gauss-Bonnet theorem as well as several other methods and found to be exactly equal $26 + k = 26.180\,339\,89$. The reader should attest for himself that $(26 + k)^2 = 685.4010167$ i.e. equal to the sum of all dimensions of the two and three Stein spaces [25,36].

Since energy, and in analogy with classical field such as for instance theory of elasticity, is proportional to a constant multiplied with the square of the curvature, then normalizing this constant our internal energy would be equal to the sum over the two and three Stein-spaces which is 685 because $(26 + k)^2 \cong 685$. Again in analogy to classical field we can readily write down the external work. This would be a parameter λ representing all the fundamental forces acting in our spacetime manifold multiplied with the distance which these forces have caused as a deformation. It is now up to us to define this distance which is directly related to a translation due to the change of curvature. We define a pseudo-metric for our Cantorian spacetime manifold. Our length will be the square root of this metric. The metric is positive definite as in Euclidean quantum field theory. Maybe we should mention that the expression Euclidean is a little misleading. In this context it is meant only to indicate that the signature is all positive, unlike the Minkowski metric. It dose not mean that the space is flat in this context. Now we set $x_1^2 = (\bar{\alpha}_1), = 60, x_2^2 = (\bar{\alpha}_2), = 30, X_3^2 = (\bar{\alpha}_3), = 9, x_4^2 = (\bar{\alpha}_4) = 1$. Consequently our distance is equal to the square root of the sum $\sqrt{100} = 10$. Our external work is therefore equal to 10 λ.

From conservation of energy we have a simple equation, namely: internal work equal to external work. From this equation we can now determine λ, which turns out to be the square of our curvature divided by ten. Looking closely at this result we realize immediately that our λ is nothing but half of the inverse electromagnetic fine structure constant $\bar{\alpha}_0 = 137.082039328$. As such this result reflects the paramount importance of electromagnetism, and that it can be geometrized exactly as Einstein did with gravity [1–4]. Scaling of this value using $(\phi)^n$ gives us then a substitute for the equations of motion.

4.3. The Lagrangian and the Golden Mean scaling of the action

For the remainder of our analysis it is better for our understanding that we reformulate the preceding analogy and write it in terms of Lagrangian and Action. Since we have no integration because we have summed over all Lie- and Stein-spaces, our action is identical to our Lagrangian and is equal to the sum of dimensions over the 17 Stein spaces minus the loading potential which is λ multiplied with the square root of the sum of the three coupling constants of the inverse electroweak coupling plus one which stands for the Planck coupling. We have replaced integration by summation and now we replace variations which in our case are simple differentiations by an even simpler repeated scaling. This is the crucial step in our analysis. We are de facto returning to the original Hermann Weyl Gauge theory and generate the equation of motion or equilibrium not by variation or differentiation but by multiplicative Cantorian scaling. This means nothing more than multiplying the result for λ with the Golden Mean to the power of n. Letting n now run for all positive and negative integers we can generate all the information encoded in our Cantorian manifold, as we will shortly demonstrate. Before continuing we will pause here for a short wile to explain the essence of Hermann Weyl's original Gauge theory [35].

4.4. The original Gauge theory of Weyl [35] and the dynamics of fractal spacetime theory

In his effort to geometrize electromagnetism Weyl thought to introduce a Gauge factor to compensate for the stretching of the field lines of the electromagnetic field. Einstein objected by saying he could do something similar in General Relativity because the non-integrability caused there by parallel transportation on a curved surface results in a mere change of direction. A change in direction is nothing physical. However, a change in the length is something physical. Therefore Weyl's Gauge theory could not work according to Einstein. Hermann Weyl reluctantly conceded the point. However the basic idea survived albeit in a different form. The Gauge factor was made imaginary and with modulus unity so that nothing changes. In addition that Gauge factor went into the phase of the wave function in quantum mechanics and not into the length, but the name stuck. Hermann Weyl said once to retort Einstein's criticism that if he had the choice between physical facts and beauty, he would choose beauty. Ironically he was right in a sense. By changing Einstein's smooth spacetime manifold to a Cantorian non-differential fractal manifold like ours, Hermann Weyl's Gauge Theory becomes applicable again. This is so because in a fractal Cantor Set there is no natural, a-priori God-given scale. This is by no means a weakness of the theory, it is the strength of the theory. In this way our theory was able to convert classical Gauge into simple scaling.

As an example let us show how our theory is able to generate the entire dimensional hierarchy of Heterotic string theory and a little bit more [25,35].

We start with

$$\lambda = \Phi^n(1/\alpha_0)/2.$$

Taking n from 1 to 6 we find the super string hierarchy.

In the above Table 1 we need of course to set $k = 0$ except for the 10 to get the usual classical value; however, with or without k the table nicely obeys our Fibonacci gross-law and shows the natural organic origin of our theory [25,35].

5. The mass spectrum of the standard model of elementary high energy particles

One of the very important advantages of our theory is that the masses of the different elementary particles and not only the coupling constants are derivable quantities, rather than values put in by hand based on experimental results into our equations. One of the first observations which are based on our fundamental scaling law is that the mass of every particle is related to almost any other particle by a more or less direct scaling. Let us give two examples for such relations which were incidentally known empirically since a long time. The first example relates the mass of the charged pion to the electron [25].

5.1. The scaling of electrons and pions

We start with the electron and pion. For the charged π Mesons we have [25].

$m_\pi^\pm = (2(\bar{\alpha}_o) - 1)m_e$. Setting $m_e = 0.511$ MeV one finds the correct value, namely 139.586844 MeV as found experimentally.

The second example is to calculate the mass of the neutral π-Meson $m_\pi^0 = (2(\bar{\alpha}_o) - 10)m_e$. Inserting one finds 134.987 844 MeV which is very close to the experimental value [25].

To appreciate where all this comes from and also comprehend our coming mass spectrum calculations we have to understand a fact of our space which relates charge to dimension in a direct way. In quantum physics not only electric and magnetic charges are understood as charges, gained and lost by a phase transition similar to the boiling and freezing of water, but also the mass is a charge. It is not like in classical physics where the mass is an inherent, but also mysterious property of matter. Quantum physics is different and massless particles acquire mass by symmetry-breaking mechanism for instance the Higg's field mechanism.

Since in our Cantorian manifold coupling constants and dimensions are in direct relation to one another, it follows that the mass-charge will be in direct relation with the dimension of the symmetry groups involved as well as the corresponding coupling constants. We can give a few empirical tests for these mathematical facts. Let us calculate the average mass of the charged and neutral Pion which we have just calculated. Based on the experimental results which we can find in any suitable table and rounding the figures to the nearest integer one finds $\langle m_\pi \rangle = (139 + 135)/2 = 137$ MeV. This is nothing but the inverse electromagnetic fine-structure coupling constant, gauged in MeV. Similarly one can show that the average mass of the Kaon equals [25] $\langle m_K \rangle = (m_K^\pm + m_K^0)/2 = 496$ MeV.

This is again nothing but the dimension of the E8 E8 exceptional Lie group, gauged in MeV.

Even more stunning than that is the fact that the mass of the η-Meson which is a fundamental problem in quantum field theory is given by the sum of all exceptional Lie groups gauged in MeV [34]:

$$m_\eta = \sum_{i=1}^{8} E_i = 548 \text{ MeV}.$$

This last particular formula is very revealing because it says that the mass of the η-Meson is equal to the average Pion when scaled by the four dimensionality of classical spacetime. All the preceding results are a consequence of the fact that $(\bar{\alpha}_{gs})^2 = (26 + k)^2 = (10)(\bar{\alpha}_o/2)$.

5.2. A mass formula

Having said all of that and having drawn the lessons from these initially astonishing empirical facts we are now in a better position to understand our next general mass formula. For this purpose we take the mass formula suggested first by Olive and Mountonen or for the same matter the Dyon mass formula [37,38]. A Dyon is a hypothetical particle which carries both electric and magnetic charge. Using this hypothesis one was able to set the following mass formula: $m^2 = c^2 (g^2 + e^2)$ where e is the dimensionless elementary electric charge, g is the elementary magnetic charge, c is a constant and m is a dimensionless mass [37,38].

It is an intuitively understandable move from the view point of our fractal-Cantorian spacetime theory to set our inverse dimensionless electromagnetic fine structure constant $\bar{\alpha}_o$ equal to $g^2 + e^2$. The formula is now reduced to our familiar scaling argument discussed earlier on. To obtain for instance the mass of the neutron all what we need is to take c equal $\phi^{-4}\sqrt{\bar{\alpha}}$ and find that $m = \phi^{-4}\bar{\alpha}_o = 939.574276$ MeV, which is the exact experimental mass of the neutron [25,37,38].

There is meantime an extensive body of literature about using our theory in the mass spectrum of elementary particles, due to the work of Tanaka in Japan, Crnjac in Slovenia, Alokaby in Egypt and many others where the reader could be provided with more details. In addition the theory can predict the most likely number of Higgs bosons which we could discover and how many particles there are in the standard model yet to be discovered experimentally, namely a maximum of nine more particles. Here we wanted only to give a flavour of the practical utility of our new spacetime theory while more detailed information can be found in the references [39–51]. In particular Elsevier's Science Direct has listed over 500 papers published on E-infinity theory [www.sciencedirect.com].

6. Conclusion

This paper became much longer than intended. Our aim was not to give a full mathematical derivation of anything. We only wanted to communicate the basic idea and flavour of a radically new spacetime theory. The theory is based entirely on topology and geometry which we call Cantorian. We could say that spacetime is regarded by us here as a Cantor set with a very high dimensionality. In fact its dimensionality is infinite. However the dimensionality is not simply infinite, it is also hierarchal. The dimensions have different weights and some weigh much heavier than others. Therefore the final overall sum is finite and is completely in accordance with what we expected. So, although all dimensions are equal for being dimensions, some dimensions are more equal than others, meaning less important. A Cantor Set is of course a collection of totally disjoint points and therefore discrete, but this is not ordinary discrete, they are transfinitely discrete and have the cardinality of the continuum. Thus, a Cantor point is not an ordinary point. Consequently our manifold which we composed here may be called after von Neumann a pointless manifold. This property has a far reaching consequence. For instance: all forms of classical gauge anomalies are eliminated and calculus has been replaced by combinatorics. Thus classical Gauge Theory has de facto been replaced by a Weyl Gauge equivalent to ordinary scaling. All these simplifications resulted in our ability to drive what traditionally has been put into theory by hand, namely the coupling constants and the mass spectrum of elementary particles. On our journey to obtain all these results we found a most unexpected bonus. We were able to drive the four dimensionality of spacetime from a basic assumption about the topology of elementary Cantor sets and could make the following conclusion. Spacetime is infinite dimensional and we live in its expectation value, namely 4.

We conclude this work by expressing our belief that particle physics and life is probably nothing more but nothing less than a fractal in the space of logic with which we mean the infinite totality of mathematical structures.

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References

[1] Hawking S, Israel W. 300 years of gravitation. Cambridge University Press; 1987.
[2] Kiefer C. Quantum gravity. Oxford University Press; 2004.
[3] Becker K, Becker M, Schwarz J. String theory and M-Theory. Cambridge University Press; 2007.
[4] El Naschie MS. Superstrings, knots and non-commutative geometry in $E^{(\infty)}$ space. Int J Theor Phys Gol 1998;37(12):2935–51.
[5] El Naschie MS. Topological defect in the Symplictic vacuum, anomalous positron production and gravitation instanton. Int J Mod Phys E Infinity Theory 2000;13(4):835–49.
[6] 'tHooft G. In search of the ultimate building blocks. Cambridge University Press; 1997.
[7] Sidharth BG. The Universe of fluctuations. The architecture of spacetime and the Universe. Dordrecht: Springer; 2005.
[8] Bardeen WA. Anomalies. In: Breitenlohner P, Maison D, editors. Quantum field theory. Berlin: Springer; 2000. p. 3–14.
[9] El Naschie MS. Anomalies free E-infinity from von Neumann's continuous geometry. Chaos, Solitons & Fractals 2008;38(5):1318–22.
[10] von Neumann J. Coninuous geometry. New Jersey: Princeton University Press; 1988.
[11] Connes A. Concommutative geometry. New York: Academic Press; 1994.
[12] El Naschie MS. Eliminating gauge anomalies via a 'point-less' fractal Yang-Mills theory. Chaos, Solitons & Fractals. 2008;38(5):1332–5.
[13] Ahmed NM, George Cantor – the father of Set Theory. The post graduate magazine, University of Newcastle-Upon-Tyne, England, 1 January, 2007, p.4–14 .
[14] Rucker R. Infinity and the mind. New Jersey: Princeton Press; 1995.
[15] Feder J, Aharony A. Fractals in physics. Elsevier, Amsterdam: North-Holland; 1990.
[16] El Naschie MS. Elementary prerequisites for E-infinity: recommended background reading in nonlinear dynamics, geometry and topology. Chaos, Solitons & Fractals 2006;30(3):579–605.
[17] Nottale L. Fractal space-time and microphysics. Singapore: World Scientific; 1993.
[18] Weibel P, Ord G, Rossler O, editors. Space time physics and fractality. Vienna, New York: Springer; 2005.
[19] Ji-Huan, He Beyond the 2006 physics Nobel prize for COBE. China Culture and Science Publishing, 2006, ISBN 988-97684-9-4/0.4.
[20] El Naschie MS, Rossler O, Prigogine I. Quantum mechanics, diffusion and chaotic fractals. Elsevier, Oxford: Pergamon; 1995. ISBN 008 0420273.
[21] Rovelli C. Quantum gravity. Cambridge University Press; 2004.
[22] El Naschie MS. P-adic unification of the fundamental forces and the standard model. Chaos, Solitons & Fractals 2008;38:1011–2.
[23] Witten E. Relections on the fate of spacetime. In: Callender C, Huggett N, editors. Physics meets philosophy at the Planck scale. Cambridge University Press; 2001. p. 125–37.
[24] El Naschie MS. Towards a quantum field theory without Gribov copies and similar problems. Chaos Solitons & Fractals 2008;38:936–8.
[25] El Naschie MS. A review of E-infinity theory and the mass spectrum of high energy particle physics. Chaos, Solitons & Fractals 2004;19:209–36.
[26] Sagan Hans. Space-filling curves. New York: Springer Verlag; 1994.
[27] Olsen S. The Golden Section, New York: Walker & Co.; 2006.
[28] El Naschie MS. On turbulence and complex dynamic in a fourdimensional peano-Hilbert space. J Franklin Inst 1993;330(1):183–98.
[29] El Naschie MS. On the uncertainty of Cantorian geometry and the two-slit experiment. Chaos Solitons & Fractals 1998;9(3):517–29.
[30] El Naschie MS. Yang-Mills instanton via exceptional Lie symmetry groups and E-infinity. Chaos Solitons & Fractals 2008;38:925–7.
[31] El Naschie MS. Average exceptional Lie and Coxeter group hierarchies with special reference to the standard model of high energy particle physics. Chaos Solitons & Fractals 2008;37(3):662–8.
[32] Peitgen HO, Jürgens H, Saupe D. Chaos and fractals. Berlin: Springer; 1992.
[33] El Naschie MS. The Feynman path integral and E-infinity from two-slit Gedanken experiment. Int J Nonlinear Sci Numerical Syst 2005;6(4):335–42.
[34] El Naschie MS. The internal dynamics of the exceptional Lie symmetry groups hierarchy and the coupling constants of unification. Chaos, Solitons & Fractals 2008;38:1031–8.
[35] El Naschie MS. From classical gauge theory back to Weyl scaling via E-infinity spacetime. Chaos, Solitons & Fractals 2008;38:980–5.

[36] El Naschie MS. Fuzzy multi-instanton knots in the fabric of space-time and Dirac's vacuum fluctuation. Chaos, Solitons & Fractals 2008;38(5):1260–8.
[37] El Naschie MS. Montonen-Olive duality of the mass spectrum of elementary particles via E-infinity. Int J Nonlinear Sci Numerical Simulation 2008;9(3):307–8.
[38] El Naschie MS, MS. BPS states, dualities and determining the mass of elementary particles. Chaos, Solitons & Fractals 2009;41(3):1263–5.
[40] Witten E. Einstein in Alexandria. Alexandria, Egypt: Bibliotheca-Alexandrina Publishing; 2006.
[41] Sidharth BG, Altaisky M. Frontiers of fundamental physics. New York: Kluwer Academic-Plenum Publishers; 2001.
[42] El Naschie MS. Quasi exceptional E12 Lie symmetry groups with 685 dimensions, KAC-Moody algebra and E-infinity Cantorian spacetime. Chaos, Solitons & Fractals 2008;38:990–2.
[43] Dickau J. Fractal cosmology. Wikipedia, Available from: <http://en.wikipedia.org>/wiki/fractalcosmology.
[44] Vrobel S. Fractal time. Wikipedia, Available from: <http://en.wikipedia.org>/wiki/fractaltime.
[45] Telegraph.co.uk Garrett Lisi and E8. Available from: <http://www.telegraph.co.uk>/earth/main/jhtml?xml=/earth/2007/11/14/scisurf114.xml.
[46] Thompson Essential Science Indicators – New Hot Papers Comments. Available from: <http://www.esi-topics.com>/nhp/2006/September-06-MohamedElNaschie.html.
[47] El Naschie MS. Dimensional symmetry breaking information and the arrow of time in Cantorian space. World Futures, OPA (Overseas Publishers Assoc.) 1997;49:391–400.
[48] Ji-Huan He. Transfinite Physics. China Culture and Scientific Publishing, China. ISBN 988-98846-5-8 (2008).
[49] Selvam AM. Chaotic Climate Dynamics, Luniver Press, United Kingdom. ISBN-13: 978-1-905986-07-1 (2007).
[50] Thompson Essential Science Indicators – Emerging Research Fronts Comments. Avaliable from: <http://esi-topics.com>/erf/2004/October04-Mohamed ElNaschie.html.
[51] Ambjorn J, Jurkiewicz J and Loll R. The Self-Organizing Quantum Universe. Scientific American, July, pp. 24-31 (2008). (http://www.sciam.com/article.cfm?id=the-self-organizing- quantum-universe).
[52] Marek-Crnjac L. A short history of fractal-Cantorian spacetime. Chaos, Solitons & Fractals 2009;41(5):2697–705.
[53] El Naschie MS. An irreducibly simple derivation of the Hausdorff dimension of spacetime. Chaos, Solitons & Fractals 2009;41(4):1902–4.
[54] Munroe R. The MSSM, E8, Hyperflavor E12 and E infinity TOE's compared and contrasted. Chaos, Solitons & Fractals 2009;41(3):1557–60.
[55] [55] El Naschie MS. Deriving the curvature of fractal-Cantorian spacetime from first principles. Chaos, Solitons & Fractals 2009;41(5):2259–61.
[56] Munroe R. Symplictic tiling, hyper colour and hyper flavour E12. Chaos, Solitons & Fractals 2009;41(4):2135–58.
[57] Crasmareanu M, Hretcanu C. Golden differential geometry. Chaos, Solitons & Fractals 2008;38:1229–38.
[58] Jean-Pierre Luminet. A cosmic hall of mirrors. Phys World, Sept. 2005, p. 23–8.
[59] El Naschie MS. The crystalographic space groups and Heterotic string theory. Chaos, Solitons & Fractals 2009;41(5):2282–4.