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An irreducibly simple derivation of the Hausdorff dimension of spacetime

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A B S T R A C T

Starting from the simple picture of a three-dimensional cube, we construct a four-dimensional cube hierarchy. The dimensions of infinitely concentric four-dimensional cubes naturally lead to a corresponding construction of a continued fraction converging to $D = 4.2360679$. This is exactly the Hausdorff dimension found using E-infinity theory.

1. Introduction

In numerous previous publications [1–3], the author and his collaborators derived the topological and Hausdorff dimension of spacetime which was found to be exactly 4 and $4 + \phi^3 = 4.236067977$, where $\phi = (\sqrt{5} - 1)/2$. The derivation of the two fundamental dimensions requires some non-trivial knowledge of the geometry and topology of Cantor sets and golden mean renormalization groups, which is not among the traditional syllabus of physics students [4]. For this reason, the author has thought for some time about an alternative way to reach the same results without resorting to these fundamental mathematical concepts.

In this short note, we give such an elementary derivation which depends mainly on geometrical visualization of a 4-D cube as shown at the bottom of the seventh column of Table 1, but nothing more.

2. The Hausdorff dimension of spacetime

It is quite easy to draw a three-dimensional cube on a two-dimensional sheet in the axionometric way shown in the figure of Table 1. On the contrary, a little more involvement is necessary to draw a four-dimensional cube. However, by simple analogy we could draw such a 4D cube, as shown in the figure at the bottom of column 7 of Table 1, which is a cube inside a larger cube and then connect the relevant points (see Figs. 2–4 of Ref. [5]). Now the dimension of this two "concentric" three-dimensional cube forming a four-dimensional one is clearly equal to four, $d = 4$. However, why should it be stopped here? We could slide inside this synthetic 4D cube another 4D cube. In general, it is not easy to draw this picture unless we enlarge the 4D cube sufficiently but in principle, it is extremely elementary. The question is what is the dimension of this new cube? It could not be $4 + 4 = 8$ because we are still working in the same space, only the volume is small. Nevertheless the new cube could not be simply four-dimensional. It also could not jump from 4 to 5 as we jumped from $D = 3$ to $D = 4$ because we are in this case constrained by the requirement that the 4D cube must fit inside the previous 4D cube. Thinking about the situation in sufficient depth will easily convince us that a reasonable measure for the new dimensionality is [1–5]

$$D_2 = D_1 + \frac{1}{D_1} = 4 + \frac{1}{4} = 4.25$$

but then again, why should we stop here? Why not go on to the second step and write [1–5]

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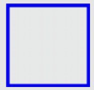
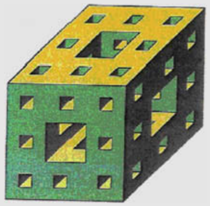
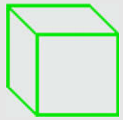
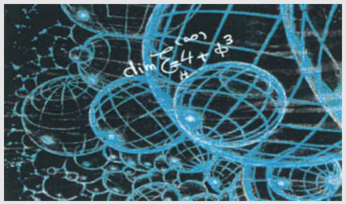
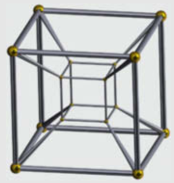
Table 1
Fractal–Hausdorff dimensions from E-infinity point of view

Type of fractal	Geometrical shape	Menger–Urysohn dimension	Hausdorff dimension	Corresponding random Hausdorff dimension	Embedding dimension	Corresponding Euclidean shape	Remark
Cantor set		0	$\ln 2/\ln 3 = 0.630929753$	$\phi = 0.61803398$	1	Line	The middle third of the line is removed and the iteration is repeated to obtain Cantor set. The final total length is zero
Sierpinski gasket		2	$\ln 3/\ln 2 = 1.584962501$	$1/\phi$=1.618033989	2	 Square	Hausdorff dimension of this fractal is the inverse of the Hausdorff dimension of the classical Cantor set. The Sierpinski is not an area. It is a 2D curve.
Menger sponge		3	$D_{MS} = \ln 20/\ln 3 = 2.7268$	$2 + \phi = 2.61803398$	3	 Cube	The COBE temperature of microwave background radiation is found to be $T_c(\text{COBE}) = D_{MS}K = 2.726$ K. The sponge is not a volume. It is a 3D curve.
The four-dimension random cantor set analogue of Menger sponge	 An artist impression of $\varepsilon^{(\infty)}$ space-time	4	$d_c^{(4)} = 4.236068$	$4 + \phi^3 = 4.23606797$	5	 Hyper cube	Note that E-infinity was not postulated but rather motivated by physical considerations. It was derived mathematically from first principles using set theory

Notice that our classification and comparison with orderly classical Cantor set, Sierpinski gasket, Menger sponge and the four-dimensional analogue of the Menger sponge and hypercube are almost complete. We just need the exact chaotic fractal shape of the fractal hypercube.

$$D_3 = D_1 + \cfrac{1}{D_1 + \cfrac{1}{D_1}}$$

$$= 4 + \cfrac{1}{4 + \cfrac{1}{4}}$$

$$= 4.235294\,.$$

The nice thing about our space is that it is infinite in both the directions, i.e. outwards and inwards. Therefore, we could continue our geometrical visualization indefinitely and find that the corresponding dimension of this "infinite" 4D cube is given by [1–5]

$$D_\infty = 4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \ldots}}}$$

$$= 4 + \left(\overline{4}\right).$$

Using a small pocket calculator, it is easily shown that $D\infty$ converges to

$$D\infty = 4.236067977.$$

The fractional part, namely 0.236067977, is easily shown to be exactly ϕ^3 where

$$\phi = (\sqrt{5} - 1)/2.$$

The reader, even if not familiar at all with our theory but has heard of fractals and fractal dimensions, will have by now surmised that $D\infty$ is the fractal dimension of our concentric 4D cubes surrounding 4D cubes and so on ad infinitum. In Table 1, we give a complete list of the classical Cantor, Sierpinski and Menger sponge in addition to the fractal four-dimensional cube which may be seen as a generalization of the Cantor set to four-dimensions. Furthermore, we give the non-classical random version of the three well known fractals obtained as in this analysis.

3. Discussion and conclusion

The fractal or more accurately the Hausdorff dimension $D = 4 + \phi^3 = (1/\phi)^3$ is indeed the expectation value of the Hausdorff dimension of our real quantum spacetime according to E-infinity theory. To understand more deeply the relation between $4 + \phi^3$, particle physics and gauge theory [6], we need to realize that it corresponds to an infinite dimensional but hierarchal Cantor set as explained in detail elsewhere. The objective of this elementary derivation was mainly to take the mystery out of a concept not that familiar in classical particle physics such as Cantor sets, Hausdorff dimensions and the like. The second recommended step is to read the extremely valuable paper by He [7,8]. After that it is quite easy to follow the more demanding papers dealing with E-infinity theory. In Table 1, we give a summary of the main ideas and results of E-infinity theory and fractal spacetime. This theory was recently the subject of renewed strong interest as can be seen from a recent article published in Scientific American and the ensuing very lively debate [9]. The author would like to thank Ayman Elokaby for help in producing Table 1.

References

[1] El Naschie MS. The VAK of vacuum fluctuation. Chaos, Solitons & Fractals 2003;18:401–20.
[2] El Naschie MS. A review of E-infinity theory and the mass spectrum of high energy particle physics. Chaos, Solitons & Fractals 2004;19:209–36.
[3] He Ji-Huan. Space, time and beyond. Int J Nonlinear Sci Numer Simulat 2005;6(4):343–6.
[4] Weibel P, Ord G, Rössler O. Spacetime physics and fractality. Festschrift in honour of Mohamed El Naschie on the occasion of his 60th birthday. Vienna, New York: Springer; 2005.
[5] El Naschie MS. Fibonacci code behind super strings and P-Branes. An answer to M. Kaku's fundamental question. Chaos Solitons & Fractals 2007;31:537–47.
[6] El Naschie MS. From classical gauge theory back to Weyl scaling via E-infinity spacetime. Chaos, Solitons & Fractals 2008;38:980–5.
[7] He Ji-Huan. Twenty-six dimensional polytope and high energy spacetime physics. Chaos, Solitons & Fractals 2007;33(1):5–13.
[8] El Naschie MS. Ji-Huan He's ten dimensional polytopes and high energy particle physics. Int J Nonlinear Sci Numer Simulat 2007;8(4):475–6.
[9] Ambjørn Jan, Jurkiewicz Jerzy, Loll Renate. The self-organizing quantum universe. Scientific American; 2008. and the online discussion at <http://www.sciam.com/article.cfm?id=the-self-organizing-quantum-universe> and The Telegraph 'Theory of Everything' by Garrett Lisi and online discussion at http://www.telegraph.co.uk/earth/main.jhtml?xml=/earth/2007/11/14/scisurf114.xml.