The VAK of vacuum fluctuation, Spontaneous self-organization and complexity theory interpretation of high energy particle physics and the mass spectrum

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Abstract

The paper is a rather informal introduction to the concepts and results of the *E*-infinity Cantorian theory of quantum physics. The fundamental tools of complexity theory and non-linear dynamics (Hausdorff dimensions, fat fractals, etc.) are used to give what we think to be a new interpretation of high energy physics and to determine the corresponding mass-spectrum. Particular attention is paid to the role played by the VAK, KAM theorem, Arnold diffusion, Newhaus sinks and knot theory in determining the stability of an elementary "particle-wave" which emerges in self-organizatory manner out of sizzling vacuum fluctuation.
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1. Basic concepts and the VAK

In the following, I would like to explain in general terms, the rationale behind my theory which is based on the conjecture that quantum space–time, i.e., the space–time seen on the extremely small scale possesses a geometry and topology which is considerably different from that of our daily experience.

It is an evident fact that space–time in our common experience is perceived to be a linearly flat smooth and continuous four-dimensional entity. This space–time which constitutes the space–time of classical mechanics possesses three space coordinates and one additional time coordinate. In other words it is a three plus one-dimensional space. We have learned from Einstein's theory that at very large velocities which are comparable to the velocity of light, time and space are fused together to a four-dimensional continuum. In addition within the extension of Einstein's theory to the general theory of relativity, this space–time continuum becomes a Reimennian non-linear curved manifold when considering the universe at large [1,2].

By contrast, within the quantum scale, we think we have many good reasons to believe that space–time looks more like a high dimensional cantor set, i.e. a disconnected transfinite point set of measure zero, which is referred to in modern parlance as a multi-dimensional fractal. This point of view has been advocated in the somewhat limited context (of deriving not only the Schrödinger and Dirac equations but also the Born postulate without quantization (deterministic or stochastic) nor analytical continuation) by the Canadian physicist Garnet Ord. Similar attempts were made later in a larger setting including cosmology by the French astrophysicist Nottale [10].

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Cantor middle third set is probably the best known and studied transfinite set. It is constructed as follows: we take a unit length and remove the middle third except for the end points so that two lines of the length of one third will remain. Then we repeat the same procedure once more with these two remaining lines. That way, we are left with four rest lines, each of the length one third square. Continuing this process ad infinitum results into the removal of infinitely many lines which added together will be as long as the unit line which we started with. In other words nothing is left except for a set of points which have no lengths, i.e. measure zero which constitute the so-called cantor set.

Nevertheless, this cantor set possesses a definite quantity that can be used to characterize it mathematically. This quantity is its Hausdorff dimension. For this particular case, that is to say, the triadic cantor sets, the Hausdorff dimension is given by the ratio of the logarithm of two divided by the logarithm of three. This comes to be approximately 0.63. This is not the only noteworthy property of a cantor set. A somewhat more surprising fact about this geometrical structure, which in a sense has long departed, is that it has the cardinality (or what is called in the terminology of George Cantor die Mächtigkeit) of the continuum.

That way transfinite set theory introduced for the first time, (via the marvelous device of Hausdorff dimension), a measure with a meaning somewhere between volume and dimension which is at the same time non-integer. In this particular case our Hausdorff dimension is telling us that the geometrical topological structure we are dealing with is something more than a point and less than a line. Consequently, it lies between the corresponding topological dimensions 0 and 1.

One can also visualize the remarkable properties of transfinite sets in the following way using the geometrical interpretation of probability theory. Let us try with an infinitely thin pencil to hit a point in the cantor set at random. The probability for being successful would be practically 0, if we think combinatorically. By contrast the same probability of the continuous line is one. From this point of view, a cantor set is nearer to being not there, that is to say nothing than to being something that is there. This is so although it is a well-defined structure with a well defined dimension. Seen however from the point of view of geometrical and topological probability, the relative probability of hitting a point should be equal to the ratio of the Hausdorff dimension of the cantor set divided by the topological dimension of the continuous line.

One could say that, from measure theoretical viewpoint, the classical cantor set does not exist. However, from the viewpoint of geometrical and topological probability cantor sets do exist. The situation may be likened to Oscar Wilde's Ghost of Canterville. My cantor sets are haunting the castle of vacuum fluctuation carrying their Hausdorff heads under their arms as a proof of their existence in a previous life.

Despite all these counter intuitive results, cantor sets do really exist mathematically as been shown and proven by George Cantor. In fact the existence of cantor sets goes far beyond being a mathematical construct. It is meantime a well known fact that the chaotic motion of classical mechanical and electromagnetic system as well as fluid flow, are intimately connected with the fractal geometry of its dynamics. And in turn these fractal geometries are nothing but the manifestation of Cantorian limit sets. Our theory will use these limit sets to revise our understanding of quantum mechanics in a way similar to what happened to our understanding of classical mechanics when deterministic chaos was rediscovered. In fact, we strongly believe that complexity theory and non-linear dynamics with KAM theorem, the VAK and Newhaus sinks [16] have a profound role to play in high-energy physics [1].

Another aspect of the remarkable properties of transfinite sets is connected to their symmetry properties. In particular, cantor sets possess scale invariance. That means they remain self-similar on all scales regardless of the resolution with which we observe or magnify any part of the set. We will see later on that this is an enormously important property reminiscent of renormalization group methods that are used in different branches of theoretical physics, particularly high energy particle physics. In fact the hidden symmetry of the Möbius transformation of space is the cause behind much of the complexity of the subatomic world of elementary particles.

To conclude this point and in anticipation of later use, we should mention that when the triadic cantor set is constructed randomly, then following a well-known theorem by the American mathematicians Mauldin and Williams, the Hausdorff dimension of the set becomes exactly equal to the golden mean. That is to say approximately 0.618 rather than the deterministic value of approximately 0.63. We will see later on that this unexpected appearance of the golden mean is a crucial point connected to KAM theorem and the fixed points of the modular groups of quantum field theories and superstrings. In addition, the same golden mean appears again in connection with the isomorphic length of Penrose–Conway tiling of space as well as with non-commutative geometry and our 4-D fusion algebra which we use in our Cantorian space–time [4]. This gives a hint about how relativity is unified with quantum mechanics via "fractal" tiling of space–time. We should mention at this point that one could easily construct a point set with positive measure. Such set is well-known in non-linear dynamics and is called Fate fractals.

There is an additional subtle point of purely theoretical nature, which one may mention on passing, to support the idea that micro-space–time must be transfinite discrete rather than a continuum. In its simplest form, this argument goes as the following: one of the most important aspects of mathematical research is undoubtedly the self-consistency of

mathematics. In order to avoid unpleasant surprises, and variants of Goedel-like theorems, mathematicians sought a rock-solid foundation for their science. That way the foundation of mathematics was reduced to a magnificent consistent structure based on the axioms of set theory. Thus set and not the continuum is the simplest thing upon which we can base our entire mathematical knowledge. Said in other words, a collection of objects rather than a continuous line is the most primitive starting point to build a theory. In short, mathematics starts with set theory and is based on it.

On the other hand, theoretical physics is clearly nothing but the application of mathematics to the objects of nature. There are extreme views for instance, those of the great man of Koenigsberg who thought a physical law has as much truthfulness in it as is expressible in mathematical terms and nothing more. In all events, theoretical physics is based on mathematics and since mathematics is based on set theory, it follows as a trivial result, that physics must be based on set theory. It is our personal conviction that this is one of a maximum handful of very few correct ways to complete Einstein's dream of geometrizing physics which converges to precisely the same point of F. Klein's Erlangen program for unifying geometry [2].

There is, however, a vital branch of mathematics that may be rivaling set theory in its application across mathematics and physics. This is measure theory and probability theory based on measure theory. Both measure theory and set theory melt into cantor transfinite set theory. It is one of the most important pillars of my theory that transfinite sets are the limit sets of group transformation of space such as the modular Möbius group transformation and that these transfinite sets are the most vital elements of the VAK which is a fractal-like attractor with a KAM cross section. The letters VAK stand for vague attractor of Kolmogorov, and KAM stands for the initials of the surnames of the three scientists who developed this theory namely, Kolmogorov, Arnold and Moser [1,16]. A schematic representation of the VAK is given in Fig. 1. In Fig. 3, we give an artist impression of *E*-infinity space.

The mere existence of something like VAK must strike one as a major surprise or even a contradiction for how could a conservative Hamiltonian system possess an attractor at all. The resolution is of course connected to the word vague and transfinite fractal geometry of the VAK as shown by Thome and others [1,2]. Incidentally, field medallist R. Thom is the man who developed catastrophe theory which was used later on for the classification of conformal field theories [8].

I would like to state that my theory proposes that the VAK in infinite dimensions, with its golden mean stability properties, is the best possible mathematical model for the stable stationary state of quantum mechanic and maybe therefore used as a model for quantum mechanical vacuum fluctuation as we will reason later on. In turn, the VAK itself will be modeled using the highly non-linear geometry and topology of our Cantorian space.

One could gain an idea about the intricate fractal dynamics involved in the VAK by looking at the fractal basin of attraction of the simple experiment shown in Ref. [15], pp. 757–765. In this particular case it is practically and theoretically impossible to predict where the motion will come to a halt for certain initial conditions although the system is completely deterministic. In the case of *E*-infinity we must stress however that non-linearity stems from the geometry and topology of space–time itself in the first place [14].

It is an almost universally accepted fact that although quite abstract, slightly isoteric and rather counter intuitive, vacuum fluctuation is the most appropriate starting point for a general theory of elementary particle. It seems therefore to be quite natural to model the counter intuitive vacuum fluctuation using the somewhat surreal geometry and topology of transfinite sets. In essence it is an extension of the same Einsteinian philosophy of geometrizing physics.
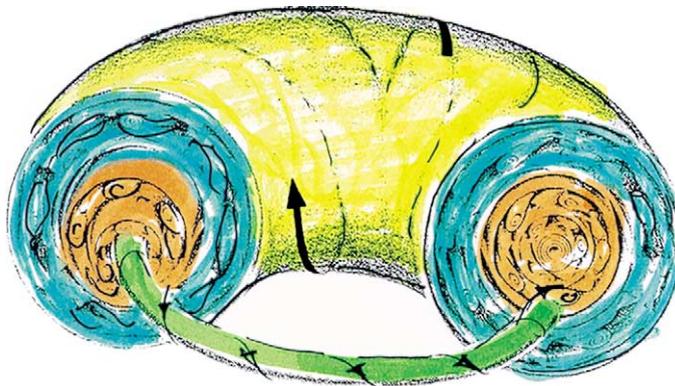

Fig. 1. A vague attractor of Kolmogorov (VAK), following Refs. [1,16]. Note that the fractal-like self-similarity of the VAK. VAK is also the goddess of vibration in the Rig-Veda which is most appropriate in the context of golden mean strings vibration.

In summing up the preceding paragraphs, I am alleging that while space–time is linearly smooth and Euclidean on our scale and curved on cosmic scales, we find that on quantum mechanical scale and as we come nearer and nearer to the Planck length, which is in the order of 10^{-33} cm, space–time acquires a cantor-set like structure, i.e. Euclidean geometry gives way to a highly non-linear fractal geometry as the form of space–time. Here the Planck length plays for space a similar but not identical role as that of the speed of light and the resulting topology is reminiscent of the wild topology of the stormy space–time foam of Wheeler [14]. An artist vision of vacuum fluctuation is shown in Fig. 2. In fact, space–time maybe defined accurately as a hierarchical infinite dimensional cantor set in which it is meaningful to speak of a quantum path of a quantum particle which has a dimensional expectation value of exactly $d = 2$ rather than the classical value $d = 1$ as shown by several authors [9]. Seen from a distance, it can be easily shown that these cantor sets can mimic the continuum and appear as if it were smooth and four-dimensional. The exact Hausdorff dimension of this space is an expectation value gained from probability theory and is exactly equal to 4 plus the golden mean to the power of 3. That is to say slightly larger than 4. The topological dimension of the embedding space by contrast is exactly 4.

In our low energy and low-resolution classical world, both the Hausdorff dimension and topological dimension coincide and space–time appears to be smooth and non-fractal and exactly four-dimensional with a classical path dimension $d = 1$ rather than the area like path dimension $d = 2$ which was discovered by Abbott and Weiss and used by Ord, Nottale and the author. The ground state of our model will be identified with the VAK which represent the
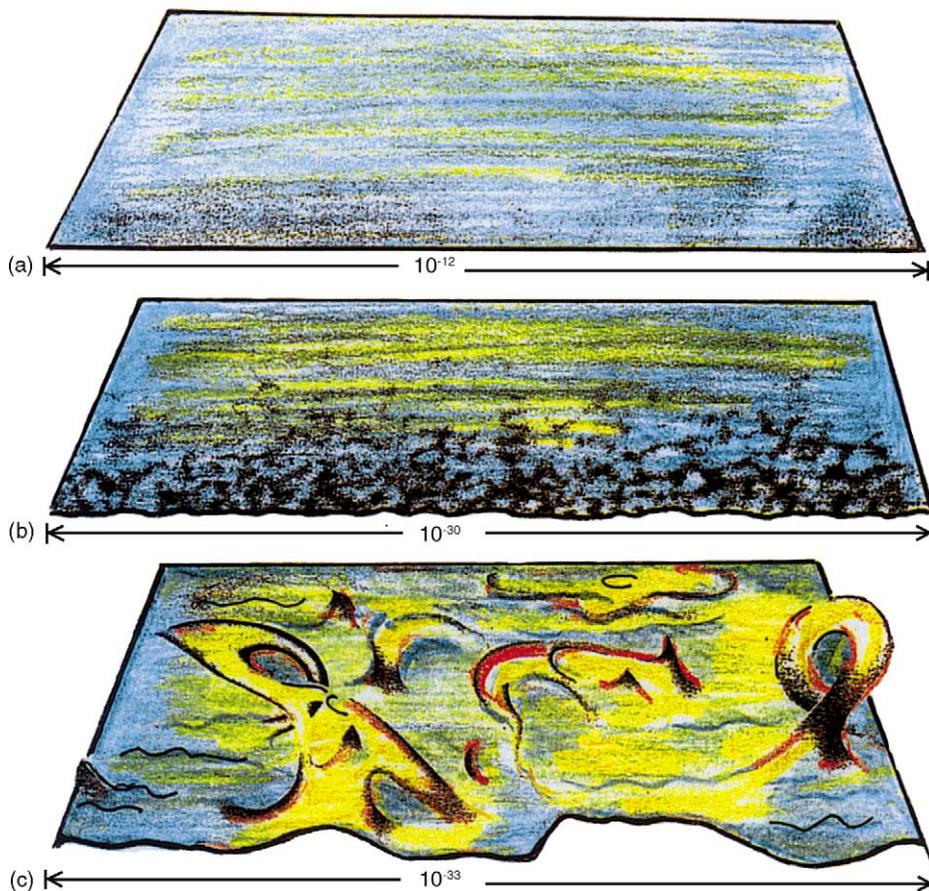

Fig. 2. A vision of the transformation of empty space from the smooth Euclidean to the rugged Planck-scale, after J.A. Wheeler's space–time foam (colouring by Sonya and Shareen El Naschie). The idea of introducing a universal length denoting the limit of measurement resolution is due to A. March (1936). W. Heisenberg attempted to abandon continuity (~1938) and used the classical radius of the electron as the smallest length $L = 10^{-13}$ cm. A. Wheeler showed that general relativity and the uncertainty principle requires the topology and the dimensionality of space–time to fluctuate violently at the Planck length $L_p = \sqrt{Gh/C^3} \cong 1.61 \times 10^{-33}$. This L_p plays a significant role in modern physical theories such as strings.

physical quantum vacuum fluctuation [1]. We also anticipate that subtle concepts such as Newhaus sinks will play a significant role in this context [1]. These are infinity of periodic orbits with extremely small basin of attraction which could be seen only at very high resolution in classical mechanics. The proximity and smallness of these sub-harmonics gives a false appearance of chaos due to the slightest amount of noise. We believe this is relevant to quantum fluctuation [1,16].

Lacking direct experimental evidence for our conjecture about the topology of quantum space–time, we think that the excellent quantitative agreement between our theoretical prediction of the fundamental constant and the mass spectrum and the experimental results is more than reasonable proof for the correctness of our model. Indeed our theory builds the universe exactly as in the title of G. Binnig's book ''Out of Nothing''. Nevertheless, we must admit that we are awaiting eagerly the discovery of the next dimension experimentally or finding a measurable deviation in Newton's law of gravitational attraction. Alternatively, it might be sufficient to measure the Hausdorff dimension of a quantum particle experimentally and find that it is larger than unity [11,12].

In the present summary, I will attempt to give also some mathematical details of the preceding description of my theory, hoping that the agreement between our theoretical prediction and some of the well-known experimental results will be persuasive enough to convince the reader of the correctness of our fundamental assumption about the nature of quantum space–time. All the same, it must be said clearly that a direct experimental verification using some ingenious experimental setup is badly needed to settle this question once and for all times [11,12].

However, before presenting any mathematical details in our present summary, we still need to clarify first our basic conception of what constitutes a particle in our ''pregeometry'' of the ''Cantorian'' space–time which we are using as well as the connection to the VAK dynamics, KAM theorem and Möbius–Klein modular groups and the appearance of the most irrational number, namely, the golden mean in all these contexts [1,2]. This we do next.

2. The string connection and KAM

One of the most successful attempts, if not the most promising of all, to go beyond the standard model, is undoubtedly superstring theory. There are various versions of this theory that we need not discuss here because what is of relevance within the present context is the main picture common to all of these variations on the string theme [8].

Our theory partially adapts the string picture of particle physics as a first approximation because this picture dispels of the infamous particle-wave duality by insisting that all elementary particles are nothing but the different modes of string vibrations [8]. These strings are commonly imagined to be of the Planck length and vibrating with the speed of light in the original light string version of the theory. In addition, these strings inhabit a 10-dimensional space–time that is dictated on us by certain consistency and anomaly cancellation requirements found by Greene and Schwartz [8]. We stress that in our theory the non-linearity of the oscillation is in the first instance due to the fractal-like geometry and wild topology of E-infinity space–time.

In our theory we go one further step and imagine these strings to have a fine structure, more precisely we conceive these strings to be made of cantor sets. In other words, in our Cantorian picture, the vibration is understood more abstractly as a symbolic dynamic of a Möbius transformation of space–time and more concretely as a sizzling violent movement of transfinite sets simulating the so-called physical vacuum fluctuation. We may mention on passing that several authors notably Crnjac and Wolfram have developed a theory of nested (fractal) vibration and obtained identical mass spectrum to mine via the Southwell and Dunkerley eigen value theorems [3]. In vibrational terms, particle physics is an incredible symphony of the microcosmos.

Thus, the VAK is almost a limit set of this dynamic and may be regarded as a valid model for vacuum fluctuation as conjectured a long time ago by the leading French topologist R.Thom. Seen from far, the sizzling cantor sets, appear as if they were the violent movement of superstrings and these strings seen from farther still appear as if they were particles [8]. In addition, we might find it occasionally useful to use Feynmann's path integral picture and take it to imply the existence of a real ''non-integrable'' quantum space–time which resembles the space–time of Ord, Nottale as well as E-infinity.

However, what about the stability of these vibration stimulating particles? The answer to this crucial stability question is found to a very far extent in KAM theory in conjunction with Arnold diffusion and Newhaus sinks [16].

The point is that our semi-classical vibrating strings including the closed heterotic strings produce a cantor dust similar to that produced by coupled chaotic maps. Consequently, the most important result of KAM theory is applicable to our situation, which means that the only string vibration ergo particles that survive a sufficiently long time to be observed are those with a sufficiently irrational winding number. Since the golden mean is the most irrational number, i.e. the irrational number that is least accurately approximated by a rational number, it follows that stable particles produced by vibrations must be related to the golden mean. Other vibrations producing other particles will not

be stable and diffuse just as fast as they are produced. It is here that the so-called Arnold diffusion is expected to play an important role. Let us first recapitulate the situation. Quantum mechanics is Hamiltonian. To substitute for the lack of friction we have the quantum. That is how Bohr made Rutherford's atom model stable.

Similarly, our semi-classical Cantorian string vibration is a conservative system. To substitute for the lack of frictional stability, we have the golden mean and to have friction losses, i.e. dissipation, we have as a substitute Arnold diffusion that causes an ergodic form of global chaos. The point is that we expect the very weak Arnold diffusion, that is almost negligible in two dimensions, to grow stronger as the number of dimensions increases indefinitely. Consequently, the smaller the distances of space–time that we probe, the stronger the influence of Arnold diffusion becomes. Again this fits perfectly in our conception of quantum vacuum fluctuation as a VAK [1]. An artist vision of the Planck length quantum gravity space–time is shown in Figs. 2 and 3.

In addition at around the Planck scale we encounter the important concept of unification of all fundamental forces. In fact, it can be shown that at a coupling constant of exactly $\bar{\alpha}_{gs} = (1/\alpha_{gs}) = (10)(1/\phi)^2 = (\bar{\alpha}_0)(\phi^2/2) = 26.18033989$ electromagnetism, all the nuclear forces and gravity become indistinguishable. We use this fundamental concept in determining the masses of the elementary particles, in particular the electron and then using scaling all other particles [1–3]. Another fundamental particle, the neutrino, we derive from the energy of the VAK.

Seen that way it comes as no surprise that the mass spectrum of all observable elementary particles that are predicted by our model are a simple or complex function of the golden mean and that these results perfectly agree with the experimental evidence. That way we have found a powerful selection rule for what can and cannot exist as a particle via KAM and Arnold diffusion. In conjunction with the dimension function of non-commutative geometry and 4-D fusion algebra, KAM and Arnold diffusion decide what particles can be observed experimentally [1–3], assuming that we have the necessary technology to do so. This interpretation implies of course that a rest mass is in reality a highly localized standing wave energy.

The final piece of the puzzle of the mass spectrum of elementary particles can be understood via the Möbius–Klein-like geometry of micro-space–time. Using this highly distortive geometry, it can be easily visualized, in an analogous way to what happened in the Beltrame Poincare projection of a hyperbolic space (see Fig. 4), that particles as a highly localized energy are various projections and distortions of each others. This might sound like a bootstrap argument but

Fig. 3. Turbulent Appollonian spheres on all scales as a model for $\varepsilon^{(\infty)}$ space or a kind of polymerized spherical micro-space–time. This is just an artistic impression and should not be taken literally.

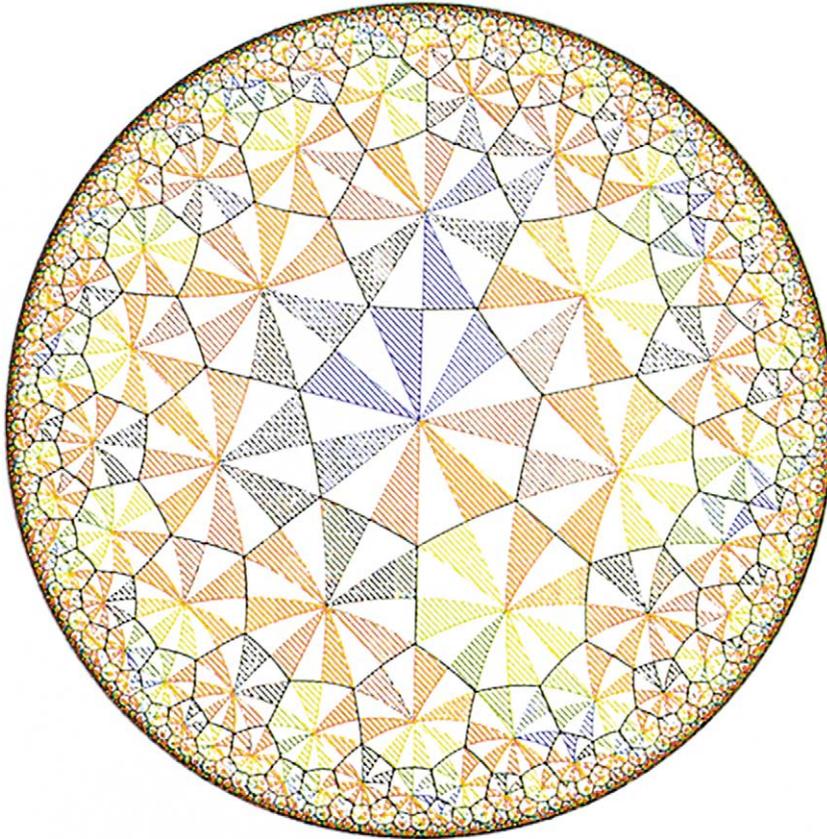Fig. 4. A hyperbolic tiling in a Beltrami–Poincare disk. The symmetries are those dictated by Möbius transformation in this case SL(2,R) and one can clearly see Klein's original modular curve inside the disk. From an intrinsically hyperbolic viewpoint, all triangles in the disk are identical. This is indirectly related to the existence of a scaling which is called the hyperbolic distance d. Note that the expectation of the normal distribution of the state of the Fischer transformation is half the value of the hyperbolic distance [32]. $E(s) = 1/2(\ln(1 + \rho)/(1 - \rho))|_{r=\rho=\phi^3} = 0.240605912$. That means $1/E(s) = \langle n \rangle = 4.15617384$. Furthermore, the inverse of the hyperbolic distance is very close to the isomorphic length of the Penrose-$E^{(\infty)}$ universe: $1/d = (\ln(1 + r)/(1 - r))^{-1} = [\ln(1/\phi)]^{-1} = 2.078086921 \simeq \ell/\rho$ where $\ell \leqslant ((4 + \phi^3)/2)\rho = (2.118033989)(\rho) \simeq (2.078086921)(\rho) \cong 1/d$. Another note worthy fact is that the constituent mass of the light quarks $m_d^* \simeq m_u^* \simeq 336$ MeV is related to the number of triangles of Klein's original curve namely $n = (2)(168) = 336$. This geometry corresponds to the internal geometry of Nonabelian guage groups.

it is not because some localized vibrations are more stable and thus more fundamental than others. For instance, the electron and the neutrino are more fundamental than the mesons.

In other words, all elementary particles seen from the energy-mass side only are in essence different scaling of any observed quasi-stable particle. This was always a well-known fact to experimentalist of high energy physics but was not taken sufficiently serious by theoreticians and consequently no systematic theoretical explanation was given before.

The scaling exponent is by contrast well understood in our model to be dictated by the scale of the region of Cantor–Klein quantum space–time that is being considered and which the particle in question inhabits. The scaling exponent is consequently, directly or indirectly related to the Hausdorff dimension of the chaotic limit set and this in turn is related to the spontaneous formation of the golden mean Möbius-trifoiles knot chains (see Fig. 5) which is reminiscent of the classical proposal of Sir W. Thomson and the modern somewhat similar concept of loop quantum mechanic [1]. We may note here parenthetically that the heterotic transfinite string is essentially a closed loop string.

In addition, we can use the preceding reasoning to ascertain that the non-existence of a particle which is geometrically admissible is basically due to the same mechanism of the observed Kirkwood gaps in the asteroid belt which again emphasizes the role of KAM as a super selection principle in high-energy physics and that VAK does indeed correspond to the stationary state of quantum mechanics. This fact was observed numerically in a recent work by C. Beck and my general theory, which he claims to have been unfortunately unaware of, is in effect the general explanation for his excellent and original numerical result which he obtained using Techebyscheff polynomial.

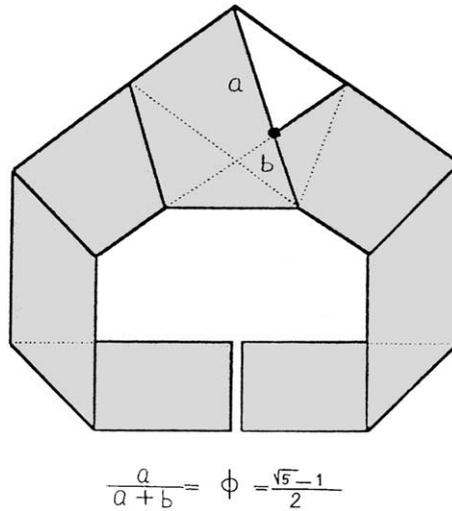$$\frac{a}{a+b} = \quad \phi \quad = \frac{\sqrt{5}-1}{2}$$

Fig. 5. An ordinary knot in a strip of paper produces a golden mean intersection of a perfect pentagon which is a part of a Möbius sling when ends are joined as indicated and the knot is made tight and carefully flattened. The corresponding theoretical knot is the well-known trefoil knot for which the hyperbolic volume is characteristically zero. Note that in the figure 8 knot possesses a hyperbolic volume equal to $V_H = 2.02988321 \simeq \langle n \rangle = 2.078086921$. This may correspond to 't Hooft monopol solution of Yang–Mills–Higgs equation.

We may mention on passing that Beck's numerical results for the fundamental coupling constants and the mass spectrum are quite close to ours but less accurate when compared to the experimental values. It is also noteworthy to mention that lacking a general theory, Beck calculated the joint probability for his results to be just a lucky coincident and found that this probability is practically zero being of the order of ten to the power of minus sixty [6].

In other words, it is impossible to attribute Beck's numerical result to anything but the fact that our Cantorian space–time theory is an accurate description of high energy particle physics [1–3]. In a nutshell, the curvature of Einstein's Riemennian geometry is the cause of gravitation and similarly, the Cantorian fractal nature of the geometry unifying both geometries is essentially the solution of quantum gravity. Thus Einstein's dream and F. Klein's program are basically identical.

3. The topology of the vacuum and the electromagnetic fine structure constant

Now we are in a position to give some mathematical details albeit very limited in scope and the reader of this summary is referred primarily to a recent paper entitled "VAK, vacuum fluctuation and the mass spectrum of high energy particle physics" [1] as well as the references therein and in particular the work of Crnjac, Ord, Koschmieder, Goldfain, Mumford, Beck, Binnig and Arnold.

We may start by a derivation of one of the most fundamental and mysterious dimensionless quantities namely, the Sommerfeld electromagnetic fine structure constant α_0 from the topology of our space–time. We should recall in this context that α_0 is a kind of geometrical probability and that $\varepsilon^{(\infty)}$ is a kind of probability space.

Let us first consider the statistical expectation value of the dimension of our E-infinity "prespace". As mentioned earlier, this space is infinite dimensional hierarchical cantor set. To arrive at our construction of $\varepsilon^{(\infty)}$, we give each of the infinitely many dimensions $n_f = \infty$ a weight $(d_c^{(0)})^n$ which is the Hausdorff dimension of a cantor set living in one dimension to the power n. The magnitude of $d_c^{(0)}$ will be the output rather than the input of our calculations so that we are not putting it by hand. Using the center of gravity theorem which constitutes a discrete gamma distribution akin to the Planck distribution discussed by El Athel [9], the expectation value of the dimensions $\sim \langle n_f \rangle$ or for short $\sim \langle n_f \rangle$ is clearly given by (see Fig. 6):

$$\sim \langle n \rangle = \frac{\sum_0^{n_f=\infty} n^2 (d_c^{(0)})^n}{\sum_0^\infty n (d_c^{(0)})^n}$$

where we can regard $(n)(d_c^{(0)})^n$ as a force and (n) as an arm of the force, so that $(n)(n)(d_c^{(0)})^n$ is a moment.
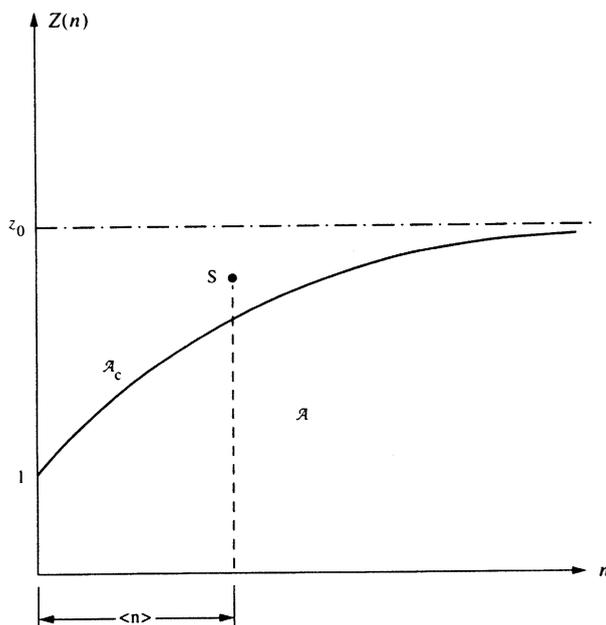
Fig. 6. Gamma space graphical interpretation of the expectation $E(n) = \langle n \rangle$ of n as the coordinate $\langle n \rangle = 2/\ln(1/d_c^{(0)})$ of the centre of gravity S of the complementary area A_c above the curve $z(n)$ which together with A forms an infinite rectangular space. In the discrete case, the curve becomes a polygon, $z(n)$ becomes $z_n = \sum_0^n (d_c^{(0)})^n$, $z_0 = \sum_0^\infty (d_c^{(0)})^n$ and $\langle n \rangle = 2/\ln(1/d_c)$ becomes $\sim \langle n \rangle = (1 + d_c^{(0)})/(1 - d_c^{(0)})$. The space $z(n)$ is called the gamma space $\Gamma(n, z_n)$ and details of the analysis can be found in [7,8].

This can be summed exactly over all dimensions in analogy to the path integral approach and Everett's many wolds interpretation of quantum mechanics and the closed form expression is a simple one:

$$\sim \langle n \rangle = (1 + d_c^{(0)})/(1 - d_c^{(0)})$$

On the other hand we know that the total sum of all the Hausdorff dimensions of all the infinitely many sets involved is given by the simple sum:

$$\sum_0^\infty (d_c^{(0)})^n = 1/(1 - d_c^{(0)})$$

Thus the relative average which has clearly the meaning of an expectation value is easily found by gauging the sum through a division by $d_c^{(0)}$. That way one finds:

$$\langle d_c \rangle = 1/[d_c^{(0)}(1 - d_c^{(0)})]$$

Incidentally this expression is identical to the dimension $[M{:}N]$ of A. Connes non-commutative geometry when setting $d_c^{(0)} = L$ and indicates that $\varepsilon^{(\infty)}$ is also non-commutative exactly as in quantum mechanics.

Our non-trivial notation may need few words of explanation. Our cantor set is described by three dimensions. First its Hausdorff dimension is written as d_c. Second this cantor set can but need not be imbedded in a line of a topological dimension $n = 1$. Finally with the Menger–Uhryson deductive dimension system a cantor has a Menger–Uhryson topological dimension zero, that is why we write $d_c^{(0)}$. Next we require from our prespace to be dimensionally consistent and since the Hausdorff dimension has also a meaning related to the volume we require that our space should have no gaps and no overlapping by insisting that both dimensions and $\sim \langle n \rangle$ should be exactly equal. Thus from $\langle d_c \rangle = \sim \langle n \rangle$ one finds [9]:

$$(1 + d_c^{(0)})/(1 - d_c^{(0)}) = 1/[(1 - d_c^{(0)})d_c^{(0)}]$$

Solving the resulting quadratic equation one finds that the only positive value is:

$$d_c^{(0)} = \phi = (\sqrt{5} - 1)/2$$

which is the golden mean. Inserting back one finds that [9]

$$\langle d_c \rangle =\sim \langle n \rangle = 4 + \phi^3 = 4.236067977$$

we note the remarkable continued fracture of $\sim \langle n \rangle$ and $\langle d_c \rangle$

$$\sim \langle n \rangle = \langle d_c \rangle = 4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cdots}}}$$

which suggests a kind of four-dimensional self-similarity. The connection between this result and 't Hooft's dimensional regularization was given in earlier papers [13]. Note that this result is obtained using a Planckian statistical distribution putting radiation and space–time on the same level. The exact correspondence between our discrete center of gravity distribution and the gamma black body distribution is of little importance in the present work and will not be given here. The interested reader may consult the literature on this point.

The essential four dimensionality of the embedding topological dimension which contains the Hausdorff dimension $4 + \phi^3$ is easily reasoned as follows. We know that lifting a Hausdorff dimension $d_c^{(0)}$ to any desirable dimension $d_c^{(n)}$ is achieved by following the bijection formula [9]:

$$d_c^{(n)} = (1/d_c^{(0)})^{n-1}$$

Thus $d_c^{(4)}$ is found using $d_c^{(0)} = \phi$ to be

$$d_c^{(4)} = (1/\phi)^{4-1} = (1 + \phi)^3 = 4 + \phi^3$$

In other words in our so-called *E*-infinity "prespace" we have

$$d_c^{(4)} =\sim \langle n \rangle = \langle d_c \rangle = 4 + \phi^3 = 4.236067977$$

while the topological dimension is exactly equal to $n_t = 4$. Thus the superscript (4) refers to the number of the topological dimension which can embed the fractal set with the Hausdorff dimension $d_c^{(4)}$.

Our physical interpretation of this situation is that in our classical low energy world of the macroobjects, the Hausdorff dimension and the topological dimension coincide and appear to be exactly 4. It is only when we probe the space at higher resolution coming near to the quantum level that we start to feel the effect of the Hausdorff dimension expectation value of $4 + \phi^3$. One could elucidate this result in slightly different manner and in analogy to Feynmann summing over all paths approach by noting that

$$\sum_0^{n_f=\infty} n\phi^n = (1)(\phi)^1 + (2)(\phi)^2 + (3)(\phi^3) + \cdots$$
$$= 4 + \phi^3$$
$$= \langle d_c \rangle$$

Now this implies that a statistical property for our space which is almost identical to that of a black body radiation is present and that $4 + \phi^3$ may be approximated by the expectation of a gamma distribution of the infinitely many hierarchical dimensions. Thus we may write approximately that (see Fig. 6)

$$\langle d_c \rangle =\sim \langle n \rangle \simeq 2/\ln(1/\phi) = 4.156173841$$

To account for the spin 1/2 of the Fermi–Dirac statistics one must write

$$\langle d_c \rangle_f = \langle d_c \rangle + 1 = 5.156173841$$

Lifting the dimension bijectively to 4*D* one finds

$$D_\alpha = [\langle d_c \rangle + 1]^{4-1}$$
$$\cong (5.156173841)^3 \cong 137.082$$

From our construction of the *E*-infinity space, we can easily reason that we are dealing indeed with a kind of a statistical geometry and topology on a random manifold. That means $\varepsilon^{(\infty)}$ is a probabilistic space. This is a direct consequence of two facts. First, the Hausdorff dimension of a random cantor set, following Mauldin–Williams theorem

is exactly equal to the golden mean. A four-dimensional random cantor set consequently possesses a Hausdorff dimension equal to our expectation value for *E*-infinity. Second, we have indeed used a statistical distribution to find our result

$$\sim \langle n \rangle = (1 + \phi)/(1 - \phi) = 4 + \phi^3$$

Thus by replacing combinatorial probability by geometrical and then topological probability where the Hausdorff dimension is replacing the volume, we have given $1/\sim \langle n \rangle$ a fundamental interpretation as being a probability. Consequently,

$$1/D_\alpha = 1/(137.082) = P_\alpha$$

can also be interpreted as a probability. In *E*-infinity we have only a set of generalized dimensions, the inverse of which are generalized probabilities. That is why we can call *E*-infinity a probabilistic 'pre'-space.

Recalling that the fine structure constant α_0 may be interpreted as cross section, i.e. a geometrical probability for an electron to absorb or emit a photon, we start our first step to connect an abstract geometrical–topological model with physics by fixing α_0 using P_α via the identification.

$$P_\alpha \equiv \alpha_0$$

This identification is at a minimum as justified as those comparable maths–physics identifications which are undertaken in many branches of theoretical physics, for instance, the Yang–Mills theory and as we will see soon, is fully justified by all the subsequent results and the entire consistency of our theory. In other words $P_\alpha \equiv \alpha_0$ is our bridge between a mathematical game $(\varepsilon^{(\infty)})$ and real physics. Only subsequent results can decide if our game is of any use in the description of reality. Deriving the mass spectrum for instance would be a demonstration of the usefulness of our identification $P_\alpha \equiv \alpha_0$.

So far we have introduced an approximate theory for α_0 using the continuous gamma distribution. The final exact result which we give here without derivation is however slightly different and states that:

$$\begin{aligned}
\bar{\alpha}_0 = 1/\alpha_0 &= (\sim \langle n \rangle)^2 (d^{(10)}) \left(1 - \frac{1}{\sim \langle n \rangle}\right) \left(1 - \frac{1}{\sim \langle n \rangle}\right) \\
&= (4 + \phi^3)^2 (10)(1 - \phi^3)(\phi^3) \\
&= 137 + \phi^5 (1 - \phi^5) \\
&= 137 + k_0 \\
&= 137.082039325 \\
&= (10)(2)(1/\phi)^4
\end{aligned}$$

The very same numerical value was found using "exact" renormalization method based on a conformal weight equation in earlier papers. Now this is the infinite dimensional *E*-infinity low energy value which is a genuine time-independent constant. This is so because in *E*-infinity of our theory time is spacialized, i.e. there is no difference between time and space unlike our $3 + 1$ space–time with time symmetry breaking introduced. Thus seen from our expectation $3 + 1$ space–time, we need to use a projection in order to cure a slight local aberration or (misfit) in a way somewhat similar to what happens with mixing angles or the isospin rotation. That way we 'project' *E*-infinity onto $3 + 1$ space in a manner of speaking. The so-obtained value is almost identical to that obtained experimentally

$$\bar{\alpha}_0(\text{Exper}) = \frac{\bar{\alpha}_0 - k_0}{\cos(\pi/\bar{\alpha}_0)} = \frac{137}{\cos(\pi/137.0820393)} = 137.03598$$

This value is slightly different from the time independent global $\bar{\alpha}_0$ which can be written in a remarkable short form based upon the multiplication and addition theorems of probability theory. This is done by interpreting $d_c^{(0)} = \phi$ as a topological probability of a Cantor set formed by the ratio of the Hausdorff dimension $d_c^{(0)} = \phi$ and the embedding topological dimension $d_c^{(1)} = 1$. That way one finds:

$$\begin{aligned}
\bar{\alpha}_0 &= 2(D^{(10)})(1/d_c^{(0)})^{D^{(4)}} \\
&= (2)(10)(1/\phi)^4 \\
&= (4)(5)(1/\phi)^4 = (20)(1/\phi)^4 = 137 + \phi^3 (1 - \phi^3)
\end{aligned}$$

We note that the so-obtained experimental value has nothing to do with running $\bar{\alpha}_0$ as a function of energy. In fact $\bar{\alpha}_0$ on the different scales can be easily computed for the electroweak, GUT and quantum gravity by 'probabilistic' or 'vibrational' combinatorics to be [1,2]

$$
\begin{aligned}
\bar{\alpha}_{ew} &= \bar{\alpha}_0 - (D^{(10)} - D^{(1)}) \\
&= (137 + k_0) - 9 \\
&= [3 + (\phi^3/10)](\bar{\alpha}_g) \\
&= [3 + (\phi^3/10)](1/\phi)(\bar{\alpha}_{gs}) \\
&= 128 + k_0 \\
&= 128.082039325
\end{aligned}
$$

$$
\begin{aligned}
\bar{\alpha}_g &= (\bar{\alpha}_0/2)(\phi) \\
&= (10)(\sim \langle n \rangle) \\
&= 42 + 2k = 42.36067977
\end{aligned}
$$

$$
\begin{aligned}
\bar{\alpha}_{gs} &= (\bar{\alpha}_0/2)(\phi^2) \\
&= (10)(1/\phi)^2 = 26 + k = 26.18033989
\end{aligned}
$$

Similarly, the expectation value for the dimension corresponding to that of the exceptional Lee group $E_8 \otimes E_8$ is given simply by [1,2,8].

$$
\text{``Dim}\, E_8 \otimes E_8 \Rightarrow (\bar{\alpha}_0)(3 + \phi) = 495.96744 \simeq 496
$$

as should be. We note that the P-Adic expansion $(137)_2 = 1 + 8 + 128$ was used in determining the above values particularly $\bar{\alpha}_{ew} = 128 = (137)_2 - 9 \cong 128.0820393$.

From determining $\bar{\alpha}_0 = 137$ (which was postulated sometime ago by Sir S. Eddington to be the space–time quasi-dimensionality or degrees of freedom required by the electron, an argument which was given some rational explanation many years later by Freeman Dyson) to determining the first mass in our mass spectrum, we have to tread a subtle road which could start at various points but arrive at the same destination. Some roads are however more mathematically sophisticated than others and we have given in the last two years different derivations of the mass spectrum via different roads.

In this context one should also mention the work of Sidharth [5] and Nottale to determine the mass spectrum which although similar in spirit, it differs considerably in methodology and scope.

4. The mass-spectrum of elementary particles and the role of space dimensions

In the present summary we could start by stating a simple physical fact as a working "physical axiom" that there is a hypothetical intermediate particle which we call the expectation π meson and that the mass of this particle is exactly equal to the inverse of the electromagnetic fine structure constant α_0 measured in MeV. This π meson, although it does not exist as such, is extremely useful because it fixes the units scale between the pure numbers of E-infinity and dimensional quantities which we measure in the laboratory. Now it is very natural to ask why should a physical fundamental quantity depend on the arbitrary choice of units? However justified this question may be, it is missing the point in our case. First of all $\langle m_\pi \rangle = \bar{\alpha}_0$ is not only a lucky mathematical choice. It is a result of calculations by the Indian physicist G. Sidharth who revived an old model that builds elementary particles out of collections of electrons and positons and the like. Using a semi classical method he reaches that way a surprisingly accurate estimate for the masses of various elementary particles. The original old model was however abandoned because it clearly lacks stability. However, by invoking the hypothesis of the structured fractal space–time, Sidharth gave the model stability and obtained the fundamental result [5].

$$
m_\pi^\pm = m_\pi^0 \simeq \bar{\alpha}_0 \simeq 137 \text{ MeV}
$$

This is however a slight simplification which he considers to be a valid approximation. Our exact result by contrast is that

$$
\langle m_\pi \rangle = \bar{\alpha}_0 \text{ MeV}
$$

from which we easily find the mass of the charged π meson to be [1,3]

$$
\begin{aligned}
m_\pi^\pm &= \bar{\alpha}_0 \text{ MeV} + [D^{(5)}/D^{(2)}] \text{ MeV} \\
&= (137 + K_0) \text{ MeV} + (5/2) \text{ MeV} \\
&= 139.58 \text{ MeV}
\end{aligned}
$$

This is almost identical to the experimental value. It may be therefore instructive to regard the choice of units as a gauge transformation which may reinforce the view of many scientists that our choice of units is crucial and should be based as far as possible on the fundamental constants of nature. In addition it seems to us that we should regard the eV system of units as being far more than a lucky choice of units.

In order to deepen our understanding of this point and to also give an example for the application of scaling transformation of E-infinity theory, we give the mass of the mesons as scaling of the mass of the electron m_e. These remarkable equations read as following:

$$
\begin{aligned}
m_\pi^\pm &= (2\bar{\alpha}_0 - 1)(m_e) \\
&= (273.1640786)(0.511) = 139.58 \text{ MeV}
\end{aligned}
$$

which perfectly agrees with our previous calculations as well as the experimental evidences. For the neutral pion, one finds

$$
\begin{aligned}
m_\pi^0 &= (2\bar{\alpha}_0 - D^{(10)})(m_e) \\
&= (2\bar{\alpha}_0 - 10)(0.511) \\
&= 134.987 \text{ MeV}
\end{aligned}
$$

Again this agrees almost exactly with the experimentally found value of

$$
m_\pi^0(\text{Exper}) = 134.98 \text{ MeV}
$$

Thus we have converted the mass of the electron to that of the charged and neutral pions using the exponents

$$
\lambda_\pi^\pm = (2\bar{\alpha}_0 - 1)
$$

and

$$
\lambda_\phi^0 = (2\bar{\alpha}_0 - 10)
$$

respectively. Remembering that $\bar{\alpha}_0$ was determined from the topology of $\varepsilon^{(\infty)}$ we see that the preceding result strongly suggests that our conjecture regarding quantum space–time is correct.

Two observations should be made immediately at this point. First, one should notice the similarity of our formulas to the dimensional function of non-commutative geometry of von Neumann and A. Connes. Second, one should remember the identification made using a simple coupled linear oscillator and 4D fusion algebra that $m_u/m_d = \phi = (\sqrt{5} - 1)/2$ where m_u and m_d are the current up and down quarks respectively. Finally, using the nested vibration model and certain Eigen value theorems, the same result may be easily obtained as shown by Crnjac [3]. Incidentally Crnjac's work is a vibrational interpretation of what may be called summing over dimensions approach in analogy to Feynmann's summing over paths in its modern version of summing over histories approach.

We can continue our scaling arguments and calculate that way numerous other particles. For instance, the mass of the neutron is particularly a neat transformation of the mass of the expectation unstable meson $\langle m_\pi \rangle$ MeV and may be written as [1–3]

$$
\begin{aligned}
m_N &= [\bar{\alpha}_0/(D^{(26)} - D^{(16)})](\langle m_\pi \rangle) \\
&= (\bar{\alpha}_0/20)(137 + k_0) \text{ MeV} \\
&= (\bar{\alpha}_0)^2/20 = 939.57427 \text{ MeV}
\end{aligned}
$$

This again is practically identical to the value found experimentally namely 939.57 MeV. The scaling in this particular case is simply

$$
\lambda_N = (\bar{\alpha}_0/20)
$$

where 20 is the first non-compactified sector of the string space, i.e. the difference between the total dimensions of the bosonic dimension $(26 + k)$ and the orbifold $(6 + k)$ of the transfinite E-infinity version of the Heterotic superstring

theory. In this context it should be mentioned that the dimensional hierarchy of the Heterotic string theory namely 26, 16, 10, 6 and 4 may be easily shown to be embedded in E-infinity by noting that [4]

$$(\bar{\alpha}_0/2)(\phi^2) = 26 + k, \quad (\bar{\alpha}_0/2)(\phi^3) = 16 + k, \quad (\bar{\alpha}_0/2)(\phi^4) = 10, \quad (\bar{\alpha}_0/2)(\phi^5) = 6 + k \quad \text{and} \quad (\bar{\alpha}_0/2)(\phi^6) = 4 - k$$

where $k = \phi^3(1 - \phi^3) = 0.180339$. For $k = 0$ we find the classical results namely 26, 16, 10, 6 and 4.

Similarly, we can treat the mass of the important K mesons. It is not difficult to see that we may define in analogy to $\langle m_\pi \rangle$ a similar quantity $\langle m_K \rangle$ where we have

$$\langle m_K \rangle = (3 + \phi)(\langle m_\pi \rangle) \text{ MeV}$$
$$= (496 - [\phi^3(1 - \phi^3)]^2) \text{ MeV}$$
$$= 495.9674775 \text{ MeV}$$

The charged and the neutral K mesons are than easily found using a simple mathematical or vibrational shift operation to be:

$$\begin{bmatrix} m_K^\pm \\ m_K^0 \end{bmatrix} = \langle m_K \rangle + \begin{bmatrix} -4/2 \\ +4/2 \end{bmatrix} = \begin{bmatrix} \langle m_K \rangle - 2 \\ \langle m_K \rangle + 2 \end{bmatrix}$$
$$= \begin{bmatrix} 493.9674775 \text{ MeV} \\ 497.9674775 \text{ MeV} \end{bmatrix}$$

in excellent agreement with the experimental measurement on K mesons. It should be noted that 496 is the dimension of the exceptional Lee group $E_8 \otimes E_8$ used in formulating the $E_8 \otimes E_8$ and the SO(32) Heterotic Superstring theory and corresponds in our theory to the value

$$496 - k^2 \quad \text{where } k = \phi^3(1 - \phi^3) = 0.18033989$$

In fact the dimension $\text{Dim} E_8 \otimes E_8 = 496$ as well as $\bar{\alpha}_0 = 137$ plays a crucial role in determining the mass spectrum of elementary particles. For instance, we could use a simple qualitative argument to give an accurate estimate to the coupling constant at the point of complete unification of all fundamental forces. The argument goes as follows: We know that the 496 generators of $E_8 \otimes E_8$ are sufficiently large to include the graviton. Since graviton is massless, it survives the very high dimensionality. This is one of the results of superstring theory which invokes 496 massless bosons instead of the mere 12 massless gauged bosons of the standard model. By analogy, the 137 dimensions may be viewed as the analogum or quasi gauge dimensions of the electromagnetic field. Consequently the square of the ratio of the two dimensions must be a measure of the coupling between these two fields and the Cooper pair charge which is the first step in a Bose condensation. In other words we can write that

$$\bar{\alpha}_{gs}/2 \simeq (496/137)^2$$
$$\simeq (3.6204375)^2$$

That means

$$\bar{\alpha}_{gs} \simeq (2)(13.10757)$$
$$\simeq 26.21$$

where $\langle d_c^{(2)} \rangle = 2$ is the Hausdorff dimension expectation value of a quantum path which is equal to the world sheet dimension of a string.

The exact expression namely $\bar{\alpha}_{gs} = 26.18033989$ is found when the transfinite correction is incorporated in our calculations. That means [4]

$$\sqrt{\frac{\bar{\alpha}_{gs}}{2}} = \left(\frac{496 - k^2}{137 + k_0} \right); \quad k_0 = \phi^5(1 - \phi^5) = 0.082039325$$

Thus

$$\bar{\alpha}_{gs} = 26.18033989 = (\bar{\alpha}_0)(\phi^2/2)$$

This is indeed the value at which all coupling constants of all fundamental forces meet in the $\bar{\alpha}_i - E$ space where E is the energy.

Using the olive-montonen duality one finds immediately the mass of the unification electron where all fundamental forces including gravity and consequently mass are indistinguishable as:

$$m_e = (\sqrt{\bar{\alpha}_{gs}})/(10)$$
$$= \left(\frac{\sqrt{2}}{10}\right)\left(\frac{496 - k^2}{137 + k_0}\right) \text{ MeV}$$
$$= 0.51166 \text{ MeV}$$

Here the fundamental particle is $\langle m_K \rangle = (496 - k^2)$ MeV and $(\sqrt{2}/10)(1/\bar{\alpha}_0) = \lambda$ is a scaling.

The mass of electron found in the laboratory is subsequently determined using projection from $\varepsilon^{(\infty)}$ onto the $4 + 1$ space which is formally the same idea of internal rotation employed previously in determining the experimental value of $\bar{\alpha}_0 = 137.036$ as following

$$m_e(\text{Exper}) = (m_e)\left(\cos\frac{\pi}{(\phi)(D^{(10)})^2}\right)$$
$$= (m_e)(\cos(\pi/61.8033989)) = 0.510999 \text{ MeV}$$

Incidentally similar to $\bar{\alpha}_{gs}$, the non-supersymmetric coupling $\bar{\alpha}_g$ may be found using $\bar{\alpha}_0$ and $\bar{\alpha}_1 = 32 + 2k$ as follows: $\bar{\alpha}_g = (\bar{\alpha}_0/\bar{\alpha}_1)(10) = 42 + 2k$ which is the exact value at which all fundamental couplings meet in non-supersymmetric case in the $\bar{\alpha}_i - E$ space where E is the energy.

This is identical to the value found in the laboratory for m_e. As in the case of $\bar{\alpha}_0$ the misfit is due to time dependency or locality which is essentially homo-morphic concept within the quantum geometry of E-infinity.

We could go on in a similar fashion and in a variety of equivalent ways determining the mass of almost all known elementary particles as well as few other particles which are not observed yet due to instabilities or lack of sufficient accuracy of observation of different sources. However, space limitation prevents us from doing that and the reader is referred to the excellent paper of Crnjac cited in Ref. [1]. Thus we conclude our present summary by giving our version of Heisenbergs isospin theory and determine the mass of the proton and finally we give our expression for the mass of the muon as a scaling of the electron using $\text{Dim} E_8 \otimes E_8$.

We have shown earlier on that the mass of the neutron is given by

$$m_N = (\bar{\alpha}_0)^2/20 = (\text{Dim} E_8 \otimes E_8 - D^{(26)})(D^{(2)}) \text{ MeV}$$

However this could be expressed in terms of the Betti number expectation

$$\bar{b}_2 = (\sqrt{\bar{\alpha}_0})(1/\phi) = \left(\frac{2m_u + m_d}{\text{MeV}}\right) = 19 - \phi^6, \quad E_8 \otimes E_8 \quad \text{and} \quad D^{(10)} = 10$$

as following:

$$m_N = ((496 - k^2)/10)(19 - \phi^6) = (\text{Dim} E_8 \otimes E_8)(\bar{b}_2/D^{(10)})$$
$$= (496 - k^2)(2m_u + m_d)/10 = 939.57 \text{ MeV}$$

Here we are assuming the neutron to consist of two up quarks and one down quark as in the classical theory of quarks. However in E-infinity theory we can derive the mass by scaling the decay product which may be a more direct and realistic theory as argued by Koschmieder, Sidharth and Crnjac.

For the proton only a minor modification is required namely replacing

$$k^2 \Rightarrow 4k$$

That way one finds the isospin internal rotation of m_N namely m_P as a projection as follows:

$$m_P = \left(\frac{496 - 4k}{10}\right)(19 - \phi^6) = 938.27 \text{ MeV}$$
$$\simeq m_N(\cos(\pi/60)) = 938.28 \text{ MeV}$$

which is identical to the experimentally found value. This is practically the same projection angle used in determining the experimental mass of the electron. In this case, the fundamental particle was taken to be $\langle m_k \rangle$ while the scaling was

$\lambda = (19 - \phi^6)/10$. A differential model using $2m_d + m_u = 21.18033989$ MeV to obtain the same result is also possible as in classical quarks theory:

$$m_P = \left(\frac{496 - k^2}{\sqrt{\bar{\alpha}_0}}\right)(22.18033989) = (m_N)\left(\cos\frac{\pi}{60}\right) \cong (2)(496 - 27) = 938.27 \text{ MeV}$$

Finally for the muon, one finds that

$$m_\mu = D^{(10)}\left(\frac{496 - k^2}{\text{Dim}\,SU(5)}\right)(m_e)$$

Noting that

$$\text{Dim}\,SU(5)|_{\varepsilon(\infty)} = [(5^2) - 1] - \phi^9 = 24 - \phi^9$$

while

$$D^{(10)} = 10 \quad \text{and} \quad m_e = 0.501 \text{ MeV}$$

one finds that

$$m_\mu = (206.7664)\ (m_e)$$
$$= 105.65766 \text{ MeV} \simeq (10^2)\left(\frac{496}{496 - 26}\right) = 105.5 \text{ MeV}$$

which is almost equal to the expectational value for m_μ. Again the fundamental particle was taken in this case to be m_e while the scaling was

$$\lambda = [(496 - k^2)/(24 - \phi^9)](10)$$

It is important to notice that the straightforward identification between the dimension of a group and the mass of a fundamental particle expressed in MeV as is the case with the average K meson

$$\langle m_K \rangle \simeq 496 \text{ MeV}$$

with

$$M_K^\pm = (\langle m_K \rangle - 2) \text{ MeV} \quad \text{and} \quad m_K^0 = (\langle m_K \rangle + 2) \text{ MeV}$$

is found for other fundamental particles. For instance we have [1,4,8]

$$\text{Dim}\,M = (1/\gamma)\left(\frac{SO(5, 21)}{SO(5) \otimes SO(21)}\right) = 105$$
$$\simeq [(\text{Dim}\,E_8 \otimes E_8)/((\text{Dim}\,E_8 \otimes E_8) - D^{(26)})] \simeq \left(\frac{496}{496 - 26}\right)(10)^2 \simeq 105$$

where M is the moduli space of string theory and consequently similar to the meson one finds that the mass of the meon is given by

$$m_\mu = (\text{Dim}\,M)\ (\text{MeV}) \simeq \left(\frac{496}{469}\right)(100) \simeq 105 \text{ MeV}$$

Similarly, for the $W^\pm$ of the electroweak we can write [4,5,8]:

$$M_W = \text{Dim Vac(moduli space)}$$
$$= [SO(4, 20)]/[SO(4) \otimes SO(20) \times T]$$
$$= 80 \text{ GeV} \simeq m_t^*/\sqrt{5} = (1 + \phi)^6(10)/\sqrt{5} = 80.25 \text{ GeV}$$

where $m_t^* = 179.4$ GeV is the constitutive mass of the top quarks.

The magical picture suggested by these relations is as if we are standing in a multi-dimensional hall of mirrors where one particle has n copies. The copies represent in the same time dimensions and these copies, being only mirror copies,

are massless particles corresponding to the number of massless gauged bosons which are nothing but the number of dimensions of our group. At symmetry breaking bifurcation the mirror hall crushes, i.e, it Bose condenses and all copies with mass exactly equal to one MeV fuse together to one particle namely the original particle which we were inferring its existence from the *n*-dimensional massless mirror copies. That way we do not need to invoke the analogous procedure of super conductivity with its two steps of Cooper pairing and Bose condensation. What I mean is that I do not need to invoke literally Higgs particles to use the field theoretical jargon.

This all sounds like Alice in the non-linearly hyperbolic quantum land but for the time being this is the best we could do toward reaching a somewhat comprehensible picture of what is the case at nuclear and sub nuclear scale of our existence where a quantum path is area-like rather than a line. Incidentally, this is a highly intriguing experimental challenge for the masters of the scientific art of experimentation [11,12].

The similar correspondence between $\varepsilon^{(\infty)}$ dimensions such as the inverse of the coupling constant $\bar{\alpha}_0$ and the average mass of the beta meson

$$\langle m_\pi \rangle = \bar{\alpha}_0 \text{ MeV}$$

is equally found for other dimensions. For instance the group $SU(3)$ has eight dimensions.

$$\text{Dim}\,SU(3) = (3^2) - 1 = 8$$

and the current mass of the fundamental down quarks is

$$\langle m_d \rangle \simeq [\text{Dim}\,SU(3)] \text{ MeV} = 8 \text{ MeV}$$

Similarly, the up quarks current mass is equal to the fermionic expectation Hausdorff dimension of *E*-infinity. That means [4].

$$\begin{aligned}\langle m_u \rangle &= (5 + \phi^3)\\ &= (\sim \langle n \rangle)(1 + \phi^3)\\ &= (\sim \langle n \rangle + 1) \text{ MeV}\end{aligned}$$

In fact the connections between pure numbers in *E*-infinity and masses of elementary particles and coupling constant goes on much further.

For instance the number of pieces in the original Klein modular curve I (see Fig. 4) is approximately equal to the mass of some of the most fundamental and elusive particles in the standard model namely the constitutive up and down quark measured in MeV $\rightarrow m_u^* \simeq m_d^* \simeq 336$ MeV.

Similarly, the experimental value of the Weinberg mixing angle of the electroweak is found from Klein's original modular curve $\Gamma(3\pi/7)$ as

$$\sin^2 \theta_W = \cos(3\pi/7) = 0.222520934$$

whereas the experimental value found from $\gamma_{\mu e} \rightarrow \gamma_{\mu e}$ is $\sin^2 \theta_\omega = 0.223 \pm 0.018$.

These are all highly interesting observations showing the undreamed of power of geometrical thinking and hinting toward some very exciting further research in this direction both theoretical and experimental.

However it is already clear at this point that the particular result regarding $m_u^* = m_d^* = 336$ MeV is a natural consequence of the relation between the number of automorphism $336/2 = 168$ of Klein's modular curve and the topology of *E*-infinity which is obvious from the relationship in the compactified case given by (see Fig. 4:)

$$\begin{aligned}(\bar{\alpha}_0/2)(\phi)(4) &= 169.442719\\ &= (\sim \langle n \rangle)(10)(4)\\ &= (\sim \langle n \rangle)(D^{(10)})(D^{(4)})\\ &\simeq 168 + \sqrt{2}\end{aligned}$$

On the other hand the geometry of the compactified version of Klein's modular curve is the geometry of the dynamics of the nested vibration interpretation of *E*-infinity theory. Conversely, the nested dynamics is nothing but the dynamics of the transfinite numbers involved in the fractal shapes of the compactified modular curve. It is here that set theory, number theory and topology meet to model the most basic dynamics in physics, namely that of the ground state vacuum of the quantum world. (see Ref. [1] and the references therein).

As a last example of the power of the geometrical picture given by the *E*-infinity theory, we may mention the determination of the mass of the electron neutrino from the cosmic background temperature energy gives $m_{ve} = [\ln 20/\ln 3](8 + \phi)(10^{-5})$ eV $\simeq 2.345(10)^{-4}$ eV, which seems to agree with the latest experimental evidence [17].

5. Spontaneous self-organization and conclusion

The present work is a very modestly detailed summary of the concepts and results of the author's *E*-infinity theory. This theory holds that quantum space–time is an infinite dimensional hierarchical cantor set and that the stationary quantum states are given by the vague attractor of Kolmogorov, the VAK. The limit set of the VAK is intimately related to the limit set of the Möbius–Klein transformation of space and may be described as a chaotic dynamical state with a highly non-linear wild topological structure (in contradistinction to the far more familiar tame topology) ramifying at a cantor set with a Hausdorff dimension related to the golden mean.

In other words, empty space is truly empty, but non-linearly structured and it is something rather than nothing although it is something very different from radiation and matter. Nevertheless it produces forces exactly as in Einstein's theory and in the spirit of the stimulating and scientifically provocative book of G. Binnig "Out of Nothingness".

However, the forces are not produced here by mere curvature of the geometry of space–time but by the chaotic juggling of its fractal geometry similar to the author's old Van der Waals analogy as proposed in one of my early papers following Feynmann [7] who was one of the first to suggest the use of fractal curves to model space–time as reported by R. Feynmann in his letter to G. Ord which was reproduced in Ref. [13].

In my theory I do not need to imagine space to be filled with invisible particles such as Higgs and then worry as we must do about the cause or the mechanism with which nature can mask the gravitational field of these particles. Using my picture, in which the conventional superstring theory can be embedded, I hope that I have shown that the quantum vacuum fluctuation can be viewed as the best starting point for a general theory of elementary particles interaction and that this fluctuation may be modeled advantageously using my Cantorian E infinity theory. In this sense, one may hope that my theory may give a waterproof understanding of Bose condensation and high temperature superconductivity and push experimental work in this direction forwards.

It should be noted that within my theory as similar to Ord's theory quantization may be thought of as a special case of the more general transfinite discretization of space–time thus unifying relativity and the quantum. As a consequence of *E*-infinity, the mass spectrum of high energy elementary particles which in conventional view is something we cannot derive theoretically but put by hand based on laboratory measurement is derived here mathematically and is found to be a function of the golden mean. The involvement of the golden mean at such fundamental level is explained here by the KAM stability requirement of the strongly localized strings-like vibration of the VAK that is capable at relatively lower resolutions to simulate particles in the sense of superstring theories. In essence there is no fundamental difference between stable particles and resonance particles. It is only the relative high stability of the first that endowed such particle with a relatively infinite life span as compared to the very short-lived resonance particle. It is the intricate interplay between KAM, Arnold diffusion and Newhaus sinks which decides on the life span of a so-called elementary particle. Similarly there is no difference between rest mass of a particle and the highly localized standing wave energy interpretation of the mass used in our *E*-infinity theory.

It is obvious that we can observe only those elementary particles produced by a stable vibration that can persist a sufficiently long time to be observed by the accuracy-dependent measuring instrument that we use. The more refined measurement we can develop the more vibration and consequently quasi particle we will be able to discover. It seems to be almost a trivial observation but it is crucial. It is all a function of hi-tech and refined experimentation technique. It is our personal opinion that determining experimentally the Hausdorff dimension of a quantum particle and finding it to be larger than one is a Nobel prize piece of work.

From a theoretical point of view, however, KAM, Arnold diffusion as well as Newhaus sinks [16] in conjunction with the conditions imposed by the dimension functions of our non-commutative geometry and the four-dimensional fusion algebra, play the role of super selection rule which determine what particles of the theoretically infinitely many possible particles may be actually realized physically and may persist sufficiently long time to be observed using our less than perfect measurement devices.

Finally it was demonstrated using many examples that all particles seen solely from the energy-mass viewpoint are different scaling of each other. In this view all particle masses are the same except for being distorted in the infinite dimensional hall of mirrors constituting our quantum space–time and which depends on the geometry and the zooming topology of the finely structured region in space–time which the concerned particle happens to be inhabiting. This general theory of scaling which parallels the same type of theory used in phase transition but at much higher energy could be given a mechanical oscillation interpretation using the eigen value theorems of Southwell and Dunkerley as shown by Marek-Crnjac [3]. In this view, particles are simply the stable combinatorial super imposition of different oscillations forming the magnificent symphony of the microcosmos which originated from a.version of what N. Wiener foresaw as a synchronization of a large number of oscillators coupled with the environment. In this case the environment is the true vacuum and the "particles" represent the spontaneous order emerging out of chaos as in complex systems, for instance the brain (see Fig. 7).

Fig. 7. Galatee Aux Sphères (1952) by Salvador Dali [© DEMART PRO ARTE BV/DACS 1996]. A mid picture singularity gives rise to an explosion of BT spheres on all scales forming globally a turbulent topological female face. A correlated chaos gives rise to form.

The magic of high energy physics, as well as what the author personally perceives as highly undesirable metaphysics, is reduced in this way to the rational Cartesian mathematical magic of the infinite dimensional geometry and topology of the magnificent transfinite sets which George Cantor foresaw with his inner eyes long before the dawn of the computer graphic age. This crystal clear and logically acceptable magic of the geometry is what we call in our low-dimensional world quantum physics at high energy.

In the final analysis the present theory is a synthesis of some pioneering work due to Einstein, Heisenberg, Dirac, Klein, Wheeler, Kaluza, Feynmann, Gross, Witten, Ord, Prigogine, Finkelstein, 't Hooft, Binnig, Smolin and many others.

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